Math 212a Lecture 2

Fejer's Theorem

For a bounded continuous function f we write $C(f,n,x)$ for the n-th Cesaro sum of its Fourier series. So if

$$a_n = a_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$$

then

$$C(f,0,x) = a_0, \quad C(f,1,x) = \frac{1}{2}(a_{-1}e^{-ix} + 2a_0 + a_1e^{ix}),$$

$$C(f,2,x) = \frac{1}{3}(a_{-2}e^{-2ix} + 2a_{-1}e^{-ix} + 3a_0 + 2a_1e^{ix} + a_2e^{2ix})$$

and, in general

$$C(f,n,x) = \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1}a_re^{irx}$$

The Fejer kernel.

$$
\begin{aligned}
C(f,n,x) &= \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} a_r e^{irx} \\
&= \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} f(t) e^{-irt} dt \right) e^{irx} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} e^{ir(x-t)} dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_n(x-t) dt, \quad \text{where} \\
K_n(s) &:= \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} e^{ins} \quad \text{so setting } y = x - t \\
C(f,n,x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) K_n(y) dy
\end{aligned}
$$

where we have used the periodicity of f and K_n.

We have

$$(e^{-is/2}+e^{is/2})^2 = e^{-is}+2+e^{is} = 2K_1(s),$$

$$(e^{-is}+1+e^{is})^2 = e^{-2is}+2e^{-is}+3+2e^{is}+e^{2is} = 3K_2(s),$$

and, in general

$$(n+1)K_n(s) = \left(\sum_{k=0}^{n} e^{i(k-\frac{n}{2})s}\right)^2 = \left(e^{-\frac{ins}{2}}\sum_{k=0}^{n} e^{iks}\right)^2.$$

If $s \neq 0 \bmod 2\pi$ we can sum this last sum as a geometric series so $(n+1)K_n(s) =$

$$\left(e^{-\frac{ins}{2}}\frac{1-e^{i(n+1)s}}{1-e^{is}}\right)^2 = \left(\frac{e^{-\frac{i(n+1)s}{2}}-e^{\frac{i(n+1)s}{2}}}{e^{-\frac{is}{2}}-e^{\frac{is}{2}}}\right)^2$$

so

$$K_n(s) = \frac{1}{n+1}\left(\frac{\sin\frac{(n+1)s}{2}}{\sin\frac{s}{2}}\right)^2$$

A closed expression for the Fejer kernel.

We have shown that

$$K_n(s) = \frac{1}{n+1}\left(\frac{\sin\frac{(n+1)s}{2}}{\sin\frac{s}{2}}\right)^2$$

for $s \neq 0 \pmod{2\pi}$ while continuity gives

$$K_n(0) = n+1.$$

$$K_n(s) \geq 0 \quad \text{for all} \quad s$$

and from its original definition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(s)ds = 1$$

since all the exponential terms integrate to zero.

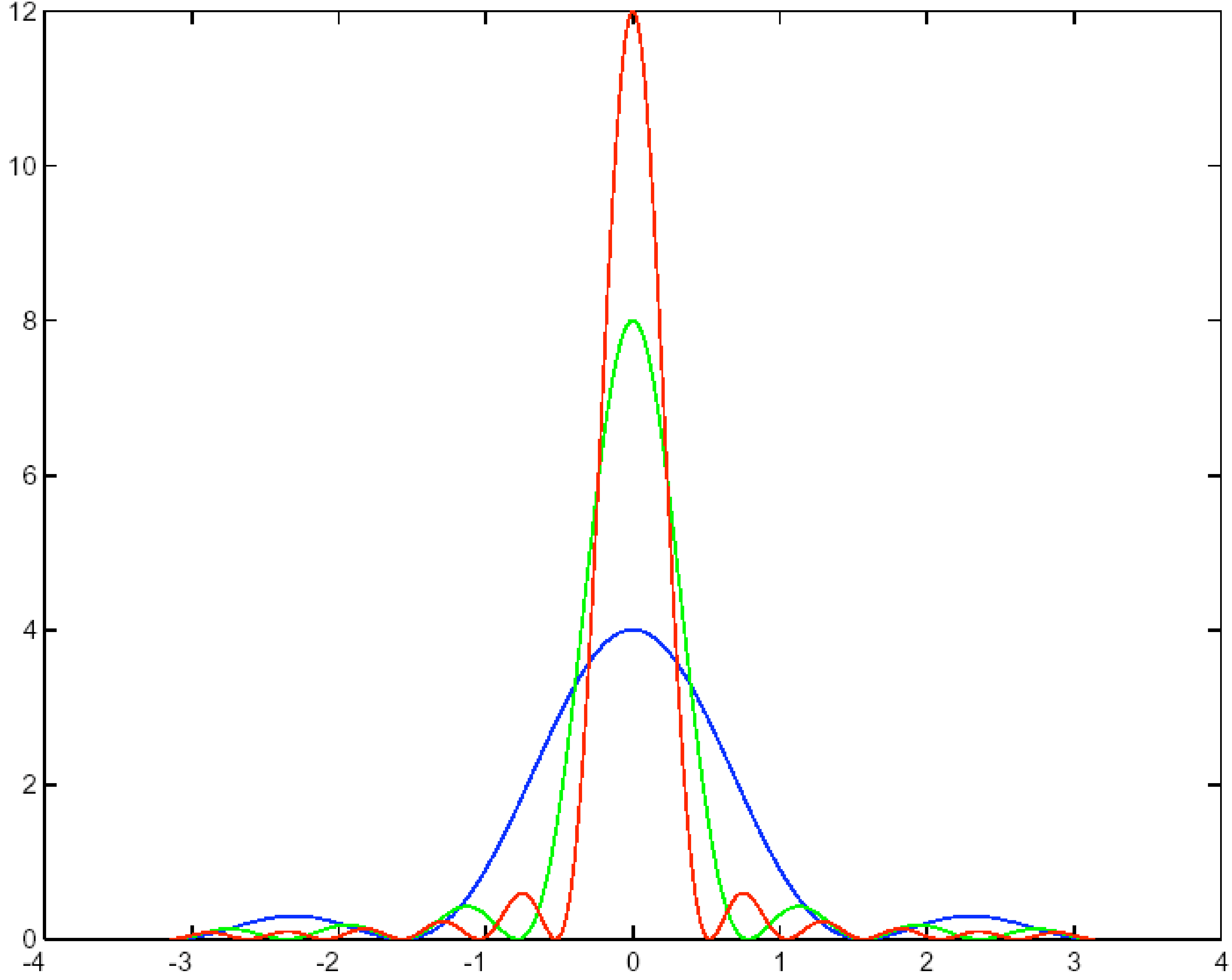


Figure 1: The graphs of K_3, K_7 and K_{11} over $[-\pi, \pi]$.

Estimating the Fejer kernel.

$$K_n(s) = \frac{1}{n+1}\left(\frac{\sin\frac{(n+1)s}{2}}{\sin\frac{s}{2}}\right)^2$$

We claim that for any $\delta > 0$ and any $\epsilon > 0$ there is an $N = N(\delta, \epsilon)$ such that

$$K_n(s) < \epsilon \quad \text{if } |s - 2\pi r| > \delta \quad \text{for all integers } r \quad \text{and } n > N.$$

Indeed, on this range, $|\sin(s/2)| > |\sin(\delta/2)| > 0$ so the denominator in the expression for K_n is bounded from below, while the numerator is bounded by 1.

Proof of Fejer's theorem, I.

$$|C(f,n,t)-f(t)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)K_n(s)ds - \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(s)f(t)ds\right| =$$

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t-s)-f(t)]K_n(s)ds\right|.$$

Now suppose that f is periodic and continuous, and let M be such that

$$|f(s)| < M$$

for all s. Then f is uniformly continuous, so, for any $\epsilon > 0$ we can find a $\delta > 0$ so that

$$|f(s) - f(t)| < \epsilon/2 \text{ if } |s-t| < \delta.$$

We can then find an N such that

$$K_n(s) < \frac{\epsilon}{4M} \text{ if } |s - 2\pi r| > \delta \text{ and } n > N.$$

Proof of Fejer's theorem, 2.

$$|C(f,n,t)-f(t)| = \quad \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t-s)-f(t)]K_n(s)ds\right|.$$

We can break this integral up into two parts. The first, over the interval $s \in [-\delta, \delta]$ is at most

$$\frac{\epsilon}{2}\frac{1}{2\pi}\int_{[-\delta.\delta]} K_n(s)ds \le \frac{\epsilon}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(s)ds = \frac{\epsilon}{2}$$

(since K_n is non-negative) while the second is at most

$$\frac{2M}{2\pi}\int_{s\notin[-\delta,\delta]} \frac{\epsilon}{4M}ds \le \frac{\epsilon}{2} \qquad \text{for } n > N \qquad .$$

So

$$|C(f,n,t)-f(t)| \le \epsilon \ \text{ if } n > N.$$

Thus we have proved Fejer's theorem which asserts that the Cesaro sum $C(f,n,t)$ of a continuous periodic function approaches $f(t)$ uniformly.

Some consequences of Fejer's theorem.

- The trigonometric polynomials are dense in the space of continuous periodic functions in the uniform topology.
- If f and g are continuous and periodic and have the same Fourier coefficients then they are equal.
- The Weierstrass approximation theorem: Any continuous function on a compact interval can be uniformly approximated by polynomials. (Since we can extend the function to be periodic, then approximate the extended function by trigonometric polynomials, and then use the Taylor series of each exponential to approximate by polynomials.)

Lipót Fejér

Born: 9 Feb 1880 in Pécs, Hungary
Died: 15 Oct 1959 in Budapest, Hungary

Dirichlet's theorem.

Dirichlet's theorem asserts that the Fourier series of a piecewise differentiable function f converges at all points x to $\frac{1}{2}(f(x_+)+f(x_-))$. Consider the n-th sum $s_n = s_n(f,x)$ of the Fourier series which is given by

$$s_n(f,x) := \sum_{-n}^{n} a_k e^{ikx}, \quad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dx.$$

We can write this as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt$$

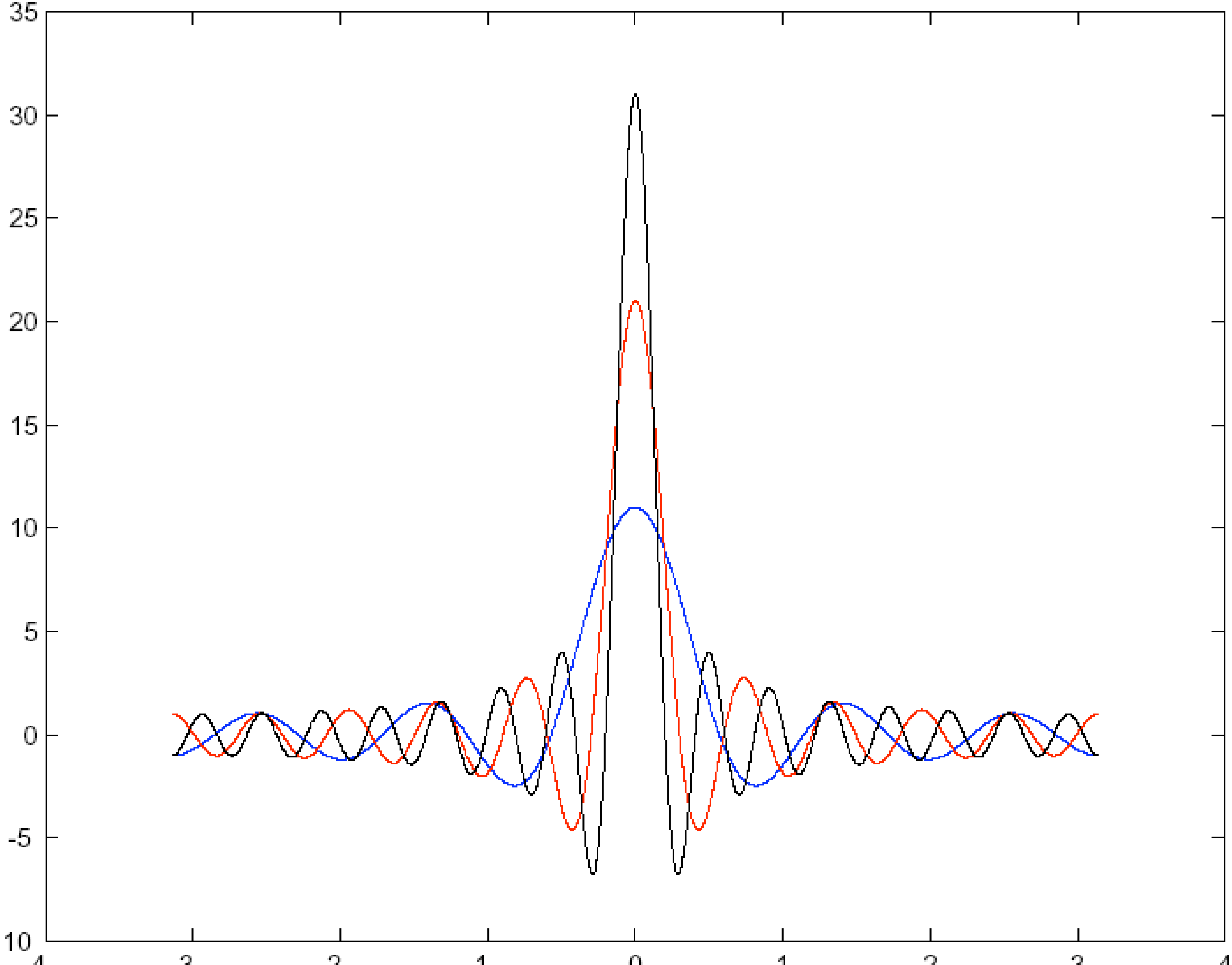
where the **Dirichlet kernel** D_n is given by

$$D_n(t) : \quad = \quad \sum_{k=-n}^{n} e^{ikt}$$

$$
\begin{aligned}
D_n(t) : &= \sum_{k=-n}^{n} e^{ikt} \\
&= e^{-int}\left(1 + e^{it} + \cdots + e^{2int}\right) \\
&= e^{-int} \cdot \frac{1 - e^{(2n+1)it}}{1 - e^{it}} \quad \text{by the sum of a geometric series} \\
&= \frac{e^{-int} - e^{i(n+1)t}}{1 - e^{it}} \\
&= \frac{e^{-i(n+\frac{1}{2})t} - e^{i(n+\frac{1}{2})t}}{e^{-i\frac{1}{2}t} - e^{i\frac{1}{2}t}} \\
&= \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}
\end{aligned}
$$

and where the value at $t = 0$ is $2n + 1$. Clearly

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)dt = 1$$



The Dirichlet kernel D_k for k= 5,10,15

Properties of the Dirichlet kernel.

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)dt = 1$$

As n increases, D_n becomes more and more oscillatory outside any fixed interval about the origin, but its amplitude does not tend to zero there, in contrast to the Fejer kernels. The issue of convergence of

$$\frac{1}{2\pi}\int_{\pi}^{\pi} f(t)D_n(x-t)dy \to f(x)$$

is more subtle.

The Riemann-Lebesgue lemma, I.

We consider functions f defined on $\mathbb{R}$ which are integrable and have the property that for any $\epsilon > 0$ there is a step function g such that

$$\int_{\mathbb{R}} |f - g| dx < \epsilon.$$

This definition depends on our definition of "integrable", and eventually, when we study the Lebesgue integral, our definition will be such that all functions which are Lebesgue integrable will have this property. Clearly, any f which is piecewise continuous and vanishes outside a finite interval has this property. Also, if f is only defined on some interval $[a, b]$ and has this property there, we just extend f by declaring it to be zero outside $[a, b]$ and the extended function belongs to our class. For the present, the Riemann integral will do. But let us denote this class of functions by $L_1(\mathbb{R})$.

The Riemann-Lebesgue lemma, 2.

A bounded integrable function h is said to satisfy the **averaging condition** if

$$\lim_{c\to\pm\infty} \frac{1}{c}\int_0^c h(t)dt \to 0.$$

For example, the function $h(t) = \sin t$ satisfies this condition.

Theorem 1 Riemann-Lebesgue Lemma. *If $f \in L_1(\mathbb{R})$ and h satisfies the averaging condition then*

$$\lim_{\omega\to\infty} \int_a^b f(t)h(\omega t)dt = 0$$

for any interval $[a, b]$.

Proof. Clearly it is enough to prove this for $a = 0, b = \infty$. If we take $f = \mathbf{1}_{[c,d]}$ where $0 \le c \le d < \infty$ then

$$\int_0^\infty f(t)h(\omega t)dt = \int_c^d h(\omega t)dt = \frac{1}{\omega}\int_0^{d\omega} h(x)dx - \frac{1}{\omega}\int_0^{c\omega} h(x)dx \to 0.$$

The Riemann-Lebesgue lemma, 3.

By linearity, the theorem is then true for step functions. Choose C such that $|h(x)| \leq C$ for all x. Let $f \in L_1(\mathbb{R})$. Choose a step function g such that

$$\int_{\mathbb{R}} |f - g| dx < \frac{\epsilon}{2C}.$$

Choose Ω such that for all $\omega > \Omega$, $\quad \left| \int_0^\infty g(t)h(\omega t)dt \right| < \frac{\epsilon}{2}$. Then for $\omega > \Omega$

$$\left| \int_0^\infty f(t)h(\omega t)dt \right| \leq \int_0^\infty |f(t) - g(t)||h(t)|dt + \left| \int_0^\infty g(t)h(\omega t)dt \right|$$

$$< \frac{\epsilon}{2C} \cdot C + \frac{\epsilon}{2} = \epsilon. \quad \square$$

The Riemann-Lebesgue lemma, 4.

We now apply the Riemann-Lebesgue Lemma to conclude that if $f \in L_1([-\pi, \pi])$ and $0 < r < \pi$ then

$$\lim_{n\to\infty} \int_r^\pi f(t) D_n(t) dt = 0.$$

indeed, on this range, the denominator, $\sin \frac{t}{2}$ of $D_n(t)$ is bounded from below so

$$t \mapsto \frac{f(t)}{\sin \frac{t}{2}} \in L_1$$

and so

$$\int_r^\pi f(t) D_n(t) dt = \int_r^\pi \frac{f(t)}{\sin \frac{t}{2}} \cdot \sin(n + \frac{1}{2}) t \; dt \to 0.$$

The Riemann-Lebesgue lemma, 5.

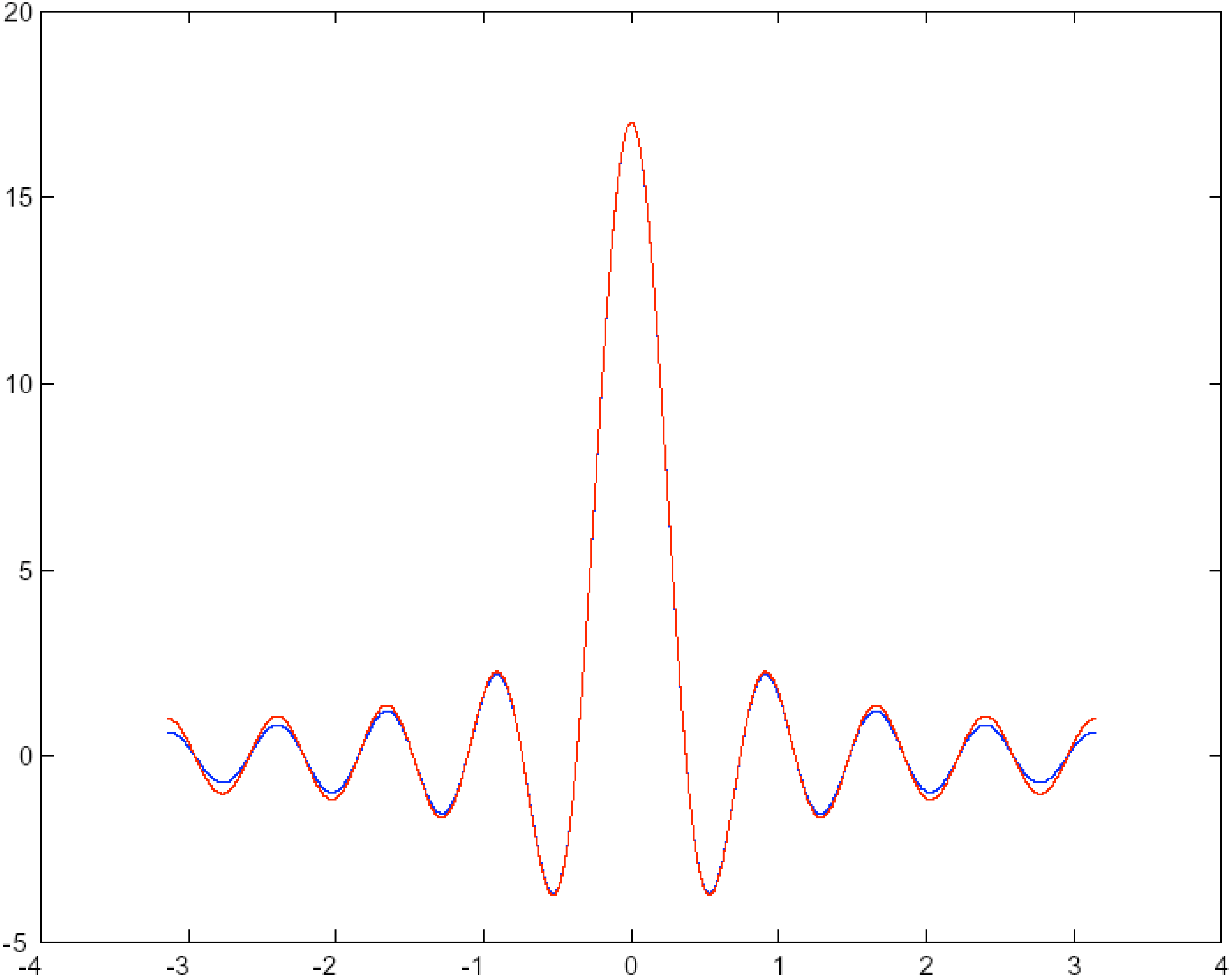
The Fourier kernel F_n is defined as

$$F_n(t) := \frac{\sin(n + \frac{1}{2})t}{\frac{t}{2}}.$$

The same argument shows that

$$\lim_{n \to \infty} \int_r^{\pi} f(t) F_n(t) dt = 0.$$

In the interval $[2, 2]$ the Fourier kernel is very close to the Dirichlet kernel for large n:



Plots of the Dirichlet (red) and Fourier (blue) kernels for k=8

The Riemann-Lebesgue lemma, 6.

Let $f \in L_1([0,\pi])$ and $0 < r < \pi$. If one or the other of the limits

$$\lim_{n\to\infty} \int_0^r f(t)D_n(t)dt \qquad \text{or} \quad \lim_{n\to\infty} \int_0^r f(t)F_n(t)dt$$

exists then both limits exist and are equal.

Proof.

$$\lim_{t\to 0} \left(\frac{1}{\sin t} - \frac{1}{t} \right) = \lim_{t\to 0} \frac{\frac{1}{3!}t^3 - \cdots}{t^2 - \cdots} = 0.$$

(More formally, apply l'Hôpital's rule twice.) Set $g(t) := \frac{1}{\sin t/2} - \frac{1}{t/2}$. So g is continuous at 0 if we set $g(0) = 0$. The difference between the two integrals is

$$\int_0^r f(t)g(t)\sin(n+\frac{1}{2})tdt$$

which tends to zero by Riemann-Lebesgue.

Proof of Dirichlet's theorem.

$$s_n(f,x) := \sum_{-n}^{n} a_k e^{ikx}, \quad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dx =$$

$$\frac{1}{2\pi}\int_{\pi}^{\pi} f(t)D_n(x-t)dt = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} f(s+x)D_n(s)ds$$

by the change of variables $s = t - x$ and the fact that D_n is even. Since both f and D_n are periodic, this last integral equals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s+x)D_n(s)ds = \frac{1}{2\pi}\int_{0}^{\pi} (f(x+s)+f(x-s))D_n(s)ds.$$

So $s_n(x) \to c$ as $n \to \infty$ if and only if there is some $r > 0$ such that

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^r [f(x+s)+f(x-s)-2c]\,\frac{\sin(n+\frac{1}{2})s}{s}ds = 0.$$

In particular, if f is differentiable from the right and left at x we get convergence to

$$c = \frac{1}{2}(f(x_+)+f(x_-))$$

which is Dirichlet's theorem.

Bessel

Although Fejer's theorem gives one interpretation of Fourier's claim that "every" periodic function can be expanded into a Fourier series, and Dirichlet's theorem gives another justification of this claim, an entirely different approach to Fourier's claim derives from the work of the astronomer Bessel, an approach derived from the method of "least squares" of great use in observational astronomy. The modern (i.e. mid 20th century) setting for this approach is the concept of a Hilbert space and the notion of an orthonormal basis.

Friedrich Wilhelm Bessel

Born: **22 July 1784 in Minden, Westphalia (now Germany)**
Died: **17 March 1846 in Königsberg, Prussia (now Kaliningrad, Russia)**

Scalar products.

V is a complex vector space. A rule assigning to every pair of vectors $f, g \in V$ a complex number (f, g) is called a **semi-scalar product** if

1. (f, g) is linear in f when g is held fixed.
2. $(g, f) = \overline{(f, g)}$. This implies that (f, g) is anti-linear in g when f is held fixed. In other words. $(f, ag + bh) = \overline{a}(f, g) + \overline{b}(f, h)$. It also implies that (f, f) is real.
3. $(f, f) \geq 0$ for all $f \in V$.

If 3. is replaced by the stronger condition

4. $(f, f) > 0$ for all non-zero $f \in V$

then we say that $(\ , \ )$ is a **scalar product**.

Examples.

- $V = \mathbf{C}^n$, so an element $\mathbf{z}$ of V is a column vector of complex numbers:
$$\mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$
and $(\mathbf{z}, \mathbf{w})$ is given by
$$(\mathbf{z}, \mathbf{w}) := \sum_1^n z_i \overline{w_i}.$$

- V consists of all continuous (complex valued) functions on the real line which are periodic of period 2π and
$$(f, g) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$
We will denote this space by $\mathcal{C}(\mathbf{T})$. Here the letter $\mathbf{T}$ stands for the one dimensional torus, i.e. the circle. We are identifying functions which are periodic with period 2π with functions which are defined on the circle $\mathbf{R}/2\pi\mathbf{Z}$.

Examples, continued.

- V consists of all doubly infinite sequences of complex numbers

$$\mathbf{a} = \ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$$

  which satisfy

$$\sum |a_i|^2 < \infty.$$

  Here

$$(\mathbf{a}, \mathbf{b}) := \sum a_i \bar{b}_i.$$

All three are examples of scalar products.

The Cauchy-Schwarz inequality.

This says that if (,) is a semi-scalar product then

$$|(f,g)| \leq (f,f)^{\frac{1}{2}}(g,g)^{\frac{1}{2}}. \tag{1}$$

Proof. For any real number t condition 3. above says that $(f - tg, f - tg) \geq 0$. Expanding out gives

$$0 \leq (f - tg, f - tg) = (f,f) - t[(f,g) + (g,f)] + t^2(g,g).$$

Since $(g,f) = \overline{(f,g)}$, the coefficient of t in the above expression is twice the real part of (f,g). So the real quadratic form

$$Q(t) := (f,f) - 2\mathrm{Re}(f,g)t + t^2(g.g)$$

is nowhere negative. So it can not have distinct real roots, and hence by the $b^2 - 4ac$ rule we get

$$4(\mathrm{Re}(f,g))^2 - 4(f,f)(g,g) \leq 0$$

or

$$(\mathrm{Re}(f,g))^2 \leq (f,f)(g,g). \tag{2}$$

$$(\mathrm{Re}(f,g))^2 \leq (f,f)(g,g). \tag{2}$$

This is useful and almost but not quite what we want. But we may apply this inequality to $h = e^{i\theta}g$ for any θ. Then $(h,h) = (g,g)$. Choose θ so that

$$(f,g) = re^{i\theta}$$

where $r = |(f,g)|$. Then

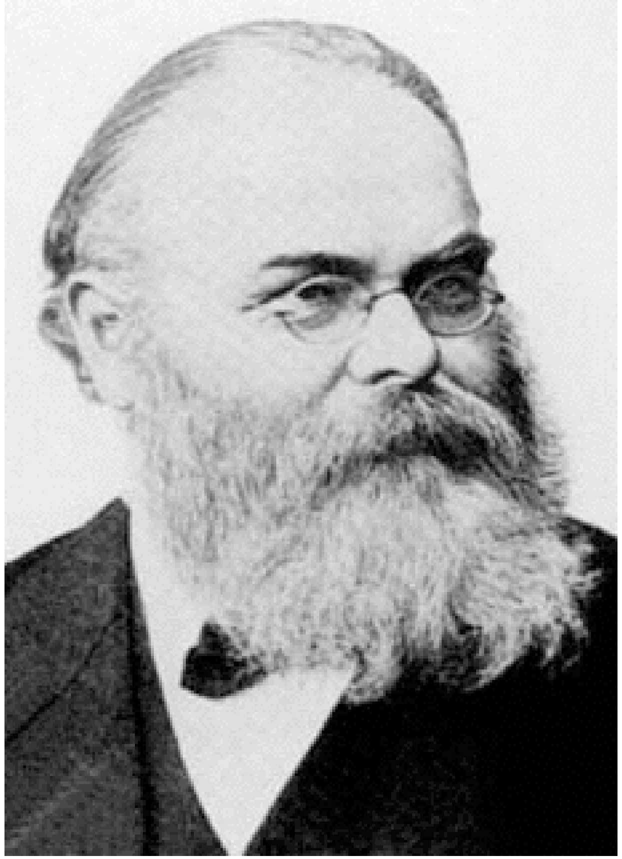
$$(f,h) = (f, e^{i\theta}g) = e^{-i\theta}(f,g) = |(f,g)|$$

and the preceding inequality with g replaced by h gives

$$|(f,g)|^2 \leq (f,f)(g,g)$$

and taking square roots gives (1).

Hermann Amandus Schwarz



Born: **25 Jan 1843 in Hermsdorf, Silesia (now Poland)**
Died: **30 Nov 1921 in Berlin, Germany**

The triangle inequality

For any semiscalar product define $\|f\| := (f,f)^{\frac{1}{2}}$

so we can write the Cauchy-Schwarz inequality as $|(f,g)| \leq \|f\|\|g\|$.

The **triangle inequality** says that

$$\|f+g\| \leq \|f\| + \|g\|. \tag{3}$$

Proof.

$$\begin{aligned}
\|f+g\|^2 &= (f+g, f+g) \\
&= (f,f) + 2\mathrm{Re}(f,g) + (g,g) \\
&\leq (f,f) + 2\|f\|\|g\| + (g,g) \quad \text{by (2)} \\
&= \|f|^2 + 2\|f\|\|g\| + \|g\|^2 \\
&= (\|f\| + \|g\|)^2.
\end{aligned}$$

Taking square roots gives the triangle inequality (3). Notice that

$$\|cf\| = |c|\|f\| \tag{4}$$

since $(cf, cf) = c\overline{c}(f,f) = |c|^2\|f\|^2$.

Pre-Hilbert spaces.

Suppose we try to define the distance between two elements of V by

$$d(f,g) := \|f-g\|.$$

Notice that then $d(f,f) = 0$, $d(f,g) = d(g,f)$ and for any three elements

$$d(f,h) \leq d(f,g) + d(g,h)$$

by virtue of the triangle inequality. The only trouble with this definition is that we might have two distinct elements at zero distance, i.e. $0 = d(f,g) = \|f-g\|$. But this can not happen if $(\ ,\ )$ is a scalar product, i.e. satisfies condition 4.

A complex vector space V endowed with a scalar product is called a **pre-Hilbert** space.

Normed spaces.

The **triangle inequality** says that

$$\|f+g\| \leq \|f\| + \|g\|. \tag{3}$$

$$\|cf\| = |c|\|f\| \tag{4}$$

Let V be a complex vector space and let $\|\cdot\|$ be a map which assigns to any $f \in V$ a non-negative real $\|f\|$ number such that $\|f\| > 0$ for all non-zero f. If $\|\cdot\|$ satisfies the triangle inequality (3) and equation (4) it is called a **norm**. A vector space endowed with a norm is called a normed space. The pre-Hilbert spaces can be characterized among all normed spaces by the parallelogram law as we will discuss below.

Later on, we will have to weaken condition (4) in our general study. But it is too complicated to give the general definition right now.

Hilbert and pre-Hilbert spaces.

The reason for the prefix "pre" is the following: The distance d defined above has all the desired properties we might expect of a distance. In particular, we can define the notions of "limit" and of a "Cauchy sequence" as is done for the real numbers: If f_n is a sequence of elements of V, and $f \in V$ we say that f is the limit of the f_n and write

$$\lim_{n\to\infty} f_n = f, \quad \text{or} \quad f_n \to f$$

if, for any positive number ϵ there is an $N = N(\epsilon)$ such that

$$d(f_n, f) < \epsilon \quad \text{for all } n \geq N.$$

If a sequence converges to some limit f, then this limit is unique, since any limits must be at zero distance and hence equal.

We say that a sequence of elements is **Cauchy** if for any $\delta > 0$ there is an $K = K(\delta)$ such that

$$d(f_m, f_n) < \delta \quad \forall, m, n \geq K.$$

If the sequence f_n has a limit, then it is Cauchy - just choose $K(\delta) = N(\frac{1}{2}\delta)$ and use the triangle inequality.

But it is quite possible that a Cauchy sequence has no limit. As an example of this type of phenomenon, think of the rational numbers with $|r - s|$ as the distance. The whole point of introducing the real numbers is to guarantee that every Cauchy sequence has a limit.

So we say that a pre-Hilbert space is a **Hilbert space** if it is "complete" in the above sense - if every Cauchy sequence has a limit.

Since the complex numbers are complete (because the real numbers are), it follows that $\mathbf{C}^n$ is complete, i.e. is a Hilbert space. Indeed, we can say that any finite dimensional pre-Hilbert space is a Hilbert space because it is isomorphic (as a pre-Hilbert space) to $\mathbf{C}^n$ for some n. (See below when we discuss orthonormal bases.)

The trouble is in the infinite dimensional case, such as the space of continuous periodic functions. This space is not complete. For example, let f_n be the function which is equal to one on $(-\pi+\frac{1}{n},-\frac{1}{n})$, equal to zero on $(\frac{1}{n},\pi-\frac{1}{n})$ and extended linearly $-\frac{1}{n}$ to $\frac{1}{n}$ and from $\pi-\frac{1}{n}$ to $\pi+\frac{1}{n}$ so as to be continuous and then extended so as to be periodic.(Thus on the interval $(\pi-\frac{1}{n},\pi+\frac{1}{n})$ the function is given by $f_n(x)=2n(x-(\pi-\frac{1}{n}))$.) If $m\le n$, the functions f_m and f_n agree outside two intervals of length $\frac{2}{m}$ and on these intervals $|f_m(x)-f_n(x)|\le 1$. So $\|f_m-f_n\|^2\le\frac{1}{2\pi}\cdot 2/m$ showing that the sequence $\{f_n\}$ is Cauchy. But the limit would have to equal one on $(-\pi,0)$ and equal zero on $(0,\pi)$ and so be discontinuous at the origin and at π. Thus the space of continuous periodic functions is not a Hilbert space, only a pre-Hilbert space.

Completing a pre-Hilbert space.

But just as we complete the rational numbers (by throwing in "ideal" elements) to get the real numbers, we may similarly complete any pre-Hilbert space to get a unique Hilbert space. See the section *Completion* in the chapter on metric spaces for a general discussion of how to complete any metric space. In particular, the completion of any normed vector space is a complete normed vector space. A complete normed space is called a **Banach** space. The general construction implies that any normed vector space can be completed to a Banach space. From the parallelogram law discussed below, it will follow that the completion of a pre-Hilbert space is a Hilbert space.

The completion of the space of continuous periodic functions will be denoted by $L^2(\mathbf{T})$.