Math 212a Lecture 6

Garding's inequality and its consequences.

The Sobolev spaces, review.

Recall that $\mathbf{T}$ now stands for the n-dimensional torus. Let $\mathbf{P} = \mathbf{P}(\mathbf{T})$ denote the space of trigonometric polynomials. These are functions on the torus of the form

$$u(x) = \sum a_\ell e^{i\ell \cdot x}$$

where

$$\ell = (\ell_1, \ldots, \ell_n)$$

is an n-tuplet of integers and the sum is finite. For each integer t (positive, zero or negative) we introduce the scalar product

$$(u, v)_t := \sum_\ell (1 + \ell \cdot \ell)^t a_\ell \overline{b}_\ell. \tag{24}$$

For $t = 0$ this is the scalar product

$$(u, v)_0 = \frac{1}{(2\pi)^n} \int_{\mathbf{T}} u(x)\overline{v(x)}dx.$$

This differs by a factor of $(2\pi)^{-n}$ from the scalar product that is used by Bers and Schecter. We will denote the norm corresponding to the scalar product $(\ ,\ )_s$ by $\|\ \|_s$.

The Laplacian, review.

If

$$\Delta := -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right)$$

the operator $(1+\Delta)$ satisfies

$$(1+\Delta)u = \sum (1+\ell\cdot\ell)a_\ell e^{i\ell\cdot x}$$

and so

$$((1+\Delta)^t u, v)_s = (u, (1+\Delta)^t v)_s = (u,v)_{s+t}$$

and

$$\|(1+\Delta)^t u\|_s = \|u\|_{s+2t}. \qquad (25)$$

We let $\mathbf{H}_t$ denote the completion of the space $\mathbf{P}$ with respect to the norm $\|\ \|_t$. Each $\mathbf{H}_t$ is a Hilbert space, and we have natural embeddings

$$\mathbf{H}_t \hookrightarrow \mathbf{H}_s \ \text{ if } s < t.$$

Equation (25) says that

$$(1+\Delta)^t : \mathbf{H}_{s+2t} \to \mathbf{H}_s$$

and is an isometry.

The generalized Cauchy-Schwarz inequality, review.

$$\|(1+\Delta)^t u\|_s = \|u\|_{s+2t}. \tag{25}$$

We then get the "generalized Cauchy-Schwarz inequality"

$$|(u,v)_s| \le \|u\|_{s+t}\|v\|_{s-t} \tag{26}$$

for any t, as a consequence of the usual Cauchy-Schwarz inequality. Indeed,

$$\begin{aligned}
\sum_\ell (1+\ell\cdot\ell)^s a_\ell \overline{b}_\ell &= \sum_\ell (1+\ell\cdot\ell)^{\frac{s+t}{2}} a_\ell (1+\ell\cdot\ell)^{\frac{s-t}{2}} \overline{b}_\ell \\
&= ((1+\Delta)^{\frac{s+t}{2}} u, (1+\Delta)^{\frac{s-t}{2}} v)_0 \\
&\le \|(1+\Delta)^{\frac{s+t}{2}} u\|_0 \|(1+\Delta)^{\frac{s-t}{2}} v\|_0 \\
&= \|u\|_{s+t}\|v\|_{s-t}.
\end{aligned}$$

From the generalized Schwarz inequality we also have a natural pairing of $\mathbf{H}_t$ with $\mathbf{H}_{-t}$ given by the extension of $(\ ,\ )_0$, so

$$|(u,v)_0| \leq \|u\|_t \|v\|_{-t}. \tag{29}$$

In fact, this pairing allows us to identify $\mathbf{H}_{-t}$ with the space of continuous linear functions on $\mathbf{H}_t$. Indeed, if ϕ is a continuous linear function on $\mathbf{H}_t$ the Riesz representation theorem tells us that there is a $w \in \mathbf{H}_t$ such that $\phi(u) = (u,w)_t$. Set

$$v := (1+\Delta)^t w.$$

Then

$$v \in \mathbf{H}_{-t}$$

and

$$(u,v)_0 = (u,(1+\Delta)^t w)_0 = (u,w)_t = \phi(u).$$

We record this fact as

$$\mathbf{H}_{-t} = (\mathbf{H}_t)^*. \tag{30}$$

Sobolev's Lemma, review.

The space $\mathbf{H}_0$ is just $L_2(\mathbf{T})$, and we can think of the space $\mathbf{H}_t$, $t > 0$ as consisting of those functions having "generalized L_2 derivatives up to order t". Certainly a function of class C^t belongs to $\mathbf{H}_t$. With a loss of degree of differentiability the converse is true:

Lemma 1 [Sobolev.] *If $u \in \mathbf{H}_t$ and*

$$t \geq \left[\frac{n}{2}\right] + k + 1$$

then $u \in C^k(\mathbf{T})$ and

$$\sup_{x \in \mathbf{T}} |D^p u(x)| \leq \text{const.} \|u\|_t \quad \textit{for } |p| \leq k. \tag{31}$$

Multiplication by a smooth function.

Suppose that ϕ is a C^∞ function on $\mathbf{T}$. Multiplication by ϕ is clearly a bounded operator on $\mathbf{H}_0 = L_2(\mathbf{T})$, and so it is also a bounded operator on $\mathbf{H}_t$, $t > 0$ since we can expand $D^p(\phi u)$ by applications of Leibnitz's rule.

For $t = -s < 0$ we know by the generalized Cauchy Schwarz inequality that

$$\|\phi u\|_t = \sup |(v, \phi u)_0| / \|v\|_s = \sup |(u, \overline{\phi} v)| / \|v\|_s \leq \|u\|_t \|\overline{\phi} v\|_s / \|v\|_s.$$

So in all cases we have

$$\|\phi u\|_t \leq (\text{const. depending on } \phi \text{ and } t) \|u\|_t. \tag{32}$$

Differential operators with smooth coefficients.

$$\|\phi u\|_t \leq (\text{const. depending on } \phi \text{ and } t)\|u\|_t. \tag{32}$$

Let

$$L = \sum_{|p| \leq m} \alpha_p(x) D^p$$

be a differential operator of degree m with C^∞ coefficients. Then it follows from the above that

$$\|Lu\|_{t-m} \leq \text{constant}\|u\|_t \tag{33}$$

where the constant depends on L and t.

Lemma 3 [Rellich's lemma.] *If $s < t$ the embedding $\mathbf{H_t} \hookrightarrow \mathbf{H}_s$ is compact.*

Proof. We must show that the image of the unit ball B of $\mathbf{H}_t$ in $\mathbf{H}_s$ can be covered by finitely many balls of radius ϵ. Choose N so large that

$$(1 + \ell \cdot \ell)^{(s-t)/2} < \frac{\epsilon}{2}$$

when $\ell \cdot \ell > N$. Let Z_t be the subspace of $\mathbf{H}_t$ consisting of all u such that $a_\ell = 0$ when $\ell \cdot \ell \leq N$. This is a space of finite codimension, and hence the unit ball of $Z_t^\perp \subset \mathbf{H}_t$ which consists of all u such that $a_\ell = 0$ when $\ell \cdot \ell > N$ can be covered by finitely many balls of radius $\frac{\epsilon}{2}$. The image of $Z_t^\perp$ in $\mathbf{H_s}$ is the orthogonal complement of the image of Z_t. On the other hand, for $u \in B \cap Z_t$ we have

$$\|u\|_s^2 \leq (1 + N)^{s-t} \|u\|_t^2 \leq \left(\frac{\epsilon}{2}\right)^2.$$

So the image of $B \cap Z_t$ is contained in a ball of radius $\frac{\epsilon}{2}$. Every element of the image of B can be written as a sum of an element in the image of $B \cap Z_t^\perp$ and an element of $B \cap Z_t$ and so the image of B is covered by finitely many balls of radius ϵ. QED

A useful inequality.

Let x, a, and b be positive numbers. Then

$$x^a + x^{-b} \geq 1$$

because if $x \geq 1$ the first summand is ≥ 1 and if $x \leq 1$ the second summand is ≥ 1. Setting $x = \epsilon^{1/a} A$ gives

$$1 \leq \epsilon A^a + \epsilon^{-b/a} A^{-b}$$

if ϵ and A are positive. Suppose that $t_1 > s > t_2$ and we set $a = t_1 - s$, $b = s - t_2$ and $A = 1 + \ell \cdot \ell$. Then we get

$$(1 + \ell \cdot \ell)^s \leq \epsilon (1 + \ell \cdot \ell)^{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} (1 + \ell \cdot \ell)^{t_2}$$

and therefore

$$\|u\|_s \leq \epsilon \|u\|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} \|u\|_{t_2} \quad \text{if } t_1 > s > t_2, \quad \epsilon > 0 \qquad (34)$$

for all $u \in \mathbf{H}_{t_1}$.

Elliptic operators.

A differential operator $L = \sum_{|p| \leq m} \alpha_p(x) D^p$ with real coefficients and m even is called **elliptic** if there is a constant $c > 0$ such that

$$(-1)^{m/2} \sum_{|p|=m} a_p(x)\xi^p \geq c(\xi \cdot \xi)^{m/2}. \tag{35}$$

In this inequality, the vector ξ is a "dummy variable". (Its true invariant significance is that it is a covector, i.e. an element of the cotangent space at x.) The expression on the left of this inequality is called the **symbol** of the operator L. It is a homogeneous polynomial of degree m in the variable ξ whose coefficients are functions of x. The symbol of L is sometimes written as $\sigma(L)$ or $\sigma(L)(x, \xi)$. Another way of expressing condition (35) is to say that there is a positive constant c such that

$$\sigma(L)(x, \xi) \geq c \quad \text{for all } x \text{ and } \xi \text{ such that } \xi \cdot \xi = 1.$$

We will assume until further notice that the operator L is elliptic and that m is a positive even integer.

Gårding's inequality.

Theorem 5 [Gårding's inequality.] *For every $u \in C^\infty(\mathbf{T})$ we have*

$$(u, Lu)_0 \geq c_1 \|u\|_{m/2}^2 - c_2 \|u\|_0^2 \qquad (36)$$

where c_1 and c_2 are constants depending on L.

Remark. If $u \in \mathbf{H}_{m/2}$, then both sides of the inequality make sense, and we can approximate u in the $\| \ \|_{m/2}$ norm by C^∞ functions. So once we prove the theorem, we conclude that it is also true for all elements of $\mathbf{H}_{m/2}$.

Strategy of the proof.

We will prove the theorem in stages:

1. When L is constant coefficient and homogeneous.
2. When L is homogeneous and approximately constant.
3. When the L can have lower order terms but the homogeneous part of L is approximately constant.
4. The general case.

Stage 1.

Ellipticity says that

$$(-1)^{m/2} \sum_{|p|=m} a_p(x)\xi^p \geq c(\xi \cdot \xi)^{m/2}. \tag{35}$$

$L = \sum_{|p|=m} \alpha_p D^p$ where the α_p are constants. Then

$$\begin{aligned}
(u, Lu)_0 &= \left(\sum a_\ell e^{i\ell\cdot x}, \sum_\ell \left(\sum_{|p|=m} \alpha_p (i\ell)^p \right) a_\ell e^{i\ell\cdot x} \right)_0 \\
&\geq c \sum_\ell (\ell \cdot \ell)^{m/2} |a_\ell|^2 \quad \text{by (35)} \\
&= c \sum [1 + (\ell \cdot \ell)^{m/2}] |a_\ell|^2 - c\|u\|_0^2 \\
&\geq cC\|u\|_{m/2}^2 - c\|u\|_0
\end{aligned}$$

where

$$C = \sup_{r \geq 0} \frac{1 + r^{m/2}}{(1+r)^{m/2}}.$$

This takes care of stage 1.

Stage 2.

$L = L_0 + L_1$ where L_0 is as in stage 1 and $L_1 = \sum_{|p|=m} \beta_p(x) D^p$ and

$$\max_{p,x} |\beta_p(x)| < \eta,$$

where η sufficiently small. (How small will be determined very soon in the course of the discussion.) We have

$$(u, L_0 u)_0 \geq c' \|u\|_{m/2}^2 - c\|u\|_0^2$$

from stage 1.

We integrate $(u, L_1 u)_0$ by parts $m/2$ times. There are no boundary terms since we are on the torus. In integrating by parts some of the derivatives will hit the coefficients. Let us collect all the these terms as I_2. The other terms we collect as I_1, so

$$I_1 = \sum \int b_{p'+p''} D^{p'} u \overline{D^{p''} u} dx$$

where $|p'| = |p''| = m/2$ and $b_r = \pm \beta_r$. We can estimate this sum by

$$|I_1| \leq \eta \cdot \text{const.} \|u\|_{m/2}^2$$

Stage 2, continued.

$$(u, L_0 u)_0 \geq c' \|u\|_{m/2}^2 - c\|u\|_0^2$$

$$I_1 = \sum \int b_{p'+p''} D^{p'} u \overline{D^{p''} u} dx \qquad |I_1| \leq \eta \cdot \text{const.} \|u\|_{m/2}^2$$

The remaining terms give a sum of the form

$$I_2 = \sum \int b_{p'q} D^{p'} u \overline{D^q u} dx$$

where $p' \leq m/2, q' < m/2$ so we have

$$|I_2| \leq \text{const.} \|u\|_{\frac{m}{2}} \|u\|_{\frac{m}{2}-1}.$$

Stage 2, continued.

$$\|u\|_s \leq \epsilon\|u\|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)}\|u\|_{t_2} \quad \text{if } t_1 > s > t_2, \quad \epsilon > 0 \qquad (34)$$

$$|I_2| \leq \text{const.}\|u\|_{\frac{m}{2}}\|u\|_{\frac{m}{2}-1}.$$

Now let us take

$$s = \frac{m}{2} - 1,\ t_1 = \frac{m}{2},\ t_2 = 0$$

in (34) which yields, for any $\epsilon > 0$,

$$\|u\|_{\frac{m}{2}-1} \leq \epsilon\|u\|_{\frac{m}{2}} + \epsilon^{-m/2}\|u\|_0.$$

Substituting this into the above estimate for I_2 gives

$$|I_2| \leq \epsilon \cdot \text{const.}\|u\|^2_{m/2} + \epsilon^{-m/2}\text{const.}\|u\|_{m/2}\|u\|_0.$$

Stage 2, concluded.

$$|I_2| \leq \epsilon \cdot \text{const.} \|u\|_{m/2}^2 + \epsilon^{-m/2} \text{const.} \|u\|_{m/2} \|u\|_0.$$

For any positive numbers a, b and ζ the inequality $(\zeta a - \zeta^{-1} b)^2 \geq 0$ implies that $2ab \leq \zeta^2 a^2 + \zeta^{-2} b^2$. Taking $\zeta^2 = \epsilon^{\frac{m}{2}+1}$ we can replace the second term on the right in the preceding estimate for $|I_2|$ by

$$\epsilon^{-m-1} \cdot \text{const.} \|u\|_0^2$$

at the cost of enlarging the constant in front of $\|u\|_{\frac{m}{2}}^2$. We have thus established that

$$|I_1| \leq \eta \cdot (\text{const.})_1 \|u\|_{m/2}^2$$

where the constant depends only on m, and

$$|I_2| \leq \epsilon (\text{const.})_2 \|u\|_{m/2}^2 + \epsilon^{-m-1} \text{const.} \|u\|_0^2$$

where the constants depend on L_1 but ϵ is at our disposal. So if $\eta (\text{const.})_1 < c'$ and we then choose ϵ so that $\epsilon (\text{const.})_2 < c' - \eta \cdot (\text{const.})_1$ we obtain Gårding's inequality for this case.

Stage 3.

Stage 3. $L = L_0 + L_1 + L_2$ where L_0 and L_1 are as in stage 2, and L_2 is a lower order operator. Here we integrate by parts and argue as in stage 2.

Stage 4.

Stage 4, the general case. Choose an open covering of T such that the variation of each of the highest order coefficients in each open set is less than the η of stage 1. (Recall that this choice of η depended only on m and the c that entered into the definition of ellipticity.) Thus, if v is a smooth function supported in one of the sets of our cover, the action of L on v is the same as the action of an operator as in case 3) on v, and so we may apply Gårding's inequality. Choose a finite subcover and a partition of unity $\{\phi_i\}$ subordinate to this cover. Write $\phi_i = \psi_i^2$ (where we choose the ϕ so that the ψ are smooth). So $\sum \psi_i^2 \equiv 1$. Now

$$(\psi_i u, L(\psi_i u))_0 \geq c'' \|\psi_i u\|_{m/2}^2 - \text{const.} \|\psi_i u\|_0^2$$

where c'' is a positive constant depending only on c, η, and on the lower order terms in L.

Stage 4, continued.

$$(\psi_i u, L(\psi_i u))_0 \geq c'' \|\psi_i u\|^2_{m/2} - \text{const.} \|\psi_i u\|^2_0$$

where c'' is a positive constant depending only on c, η, and on the lower order terms in L. We have

$$(u, Lu)_0 = \int (\sum \psi_i^2 u) \overline{Lu} dx = \sum (\psi_i u, L\psi_i u)_0 + R$$

where R is an expression involving derivatives of the ψ_i and hence lower order derivatives of u. These can be estimated as in case 2) above, and so we get

$$(u, Lu)_0 \geq c''' \sum \|\psi_i u\|^2_{m/2} - \text{const.} \|u\|^2_0 \qquad (37)$$

since $\|\psi_i u\|_0 \leq \|u\|_0$.

Conclusion of proof.

$$(u, Lu)_0 \geq c''' \sum \|\psi_i u\|_{m/2}^2 - \text{const.}\|u\|_0^2 \tag{37}$$

since $\|\psi_i u\|_0 \leq \|u\|_0$. Now $\|u\|_{m/2}$ is equivalent, as a norm, to $\sum_{p \leq m/2} \|D^p u\|_0$ as we verified in the preceding section. Also

$$\sum \|D^p(\psi_i u)\|_0 = \sum \|\psi_i D^p u\|_0 + R'$$

where R' involves terms differentiating the ψ and so lower order derivatives of u. Hence

$$\sum \|\psi_i u\|_{m/2}^2 \geq \text{pos. const.}\|u\|_{m/2}^2 - \text{const.}\|u\|_0^2$$

by the integration by parts argument again. Hence by (37)

$$(u, Lu)_0 \geq c''' \sum \|\psi_i u\|_{m/2}^2 - \text{const.}\|u\|_0^2$$

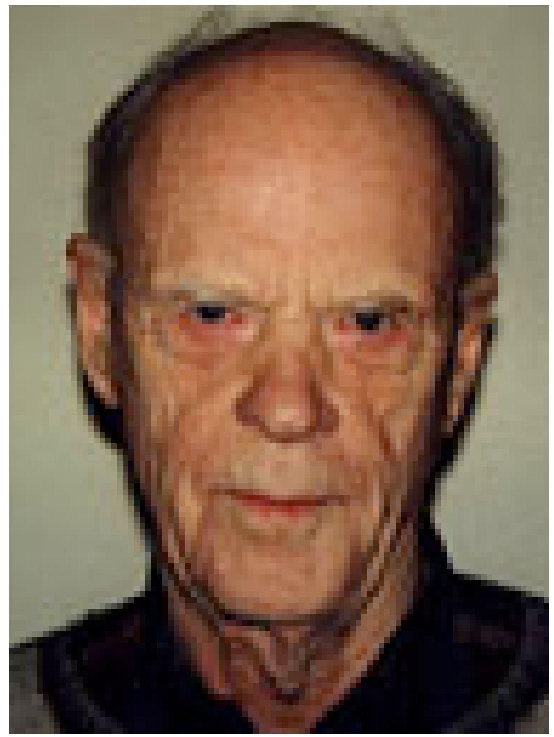
$$\geq \text{pos. const.}\|u\|_{m/2}^2 - \text{const.}\|u\|_0^2$$

which is Gårding's inequality. QED

A look ahead.

For the time being we will continue to study the case of the torus. But a look ahead is in order. In this last step of the argument, where we applied the partition of unity argument, we have really freed ourselves of the restriction of being on the torus. Once we make the appropriate definitions, we will then get Gårding's inequality for elliptic operators on manifolds. Furthermore, the consequences we are about to draw from Gårding's inequality will be equally valid in the more general setting.

Lars Gårding



Born in 1919

Consequences of Gårding's inequality.

Proposition 3 *For every integer t there is a constant $c(t) = c(t, L)$ and a positive number $\Lambda = \Lambda(t, L)$ such that*

$$\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \tag{38}$$

when

$$\lambda > \Lambda$$

for all smooth u, and hence for all $u \in \mathbf{H}_t$.

Proof. Let s be some non-negative integer. We will first prove (38) for $t = s + \frac{m}{2}$. We have

$$\|u\|_t\|Lu + \lambda u\|_{t-m} = \|u\|_t\|Lu + \lambda u\|_{s-\frac{m}{2}}$$

$$= \|u\|_t\|(1 + \Delta)^s Lu + \lambda(1 + \Delta)^s u\|_{-s-\frac{m}{2}}$$

$$\geq (u, (1 + \Delta)^s Lu + \lambda(1 + \Delta)^s u)_0$$

by the generalized Cauchy - Schwarz inequality (26).

$$\|u\|_t\|Lu+\lambda u\|_{t-m} = \|u\|_t\|Lu+\lambda u\|_{s-\frac{m}{2}}$$

$$= \|u\|_t\|(1+\Delta)^s Lu + \lambda(1+\Delta)^s u\|_{-s-\frac{m}{2}}$$

$$\geq (u, (1+\Delta)^s Lu + \lambda(1+\Delta)^s u)_0$$

by the generalized Cauchy - Schwarz inequality (26).

$$\|(1+\Delta)^t u\|_s = \|u\|_{s+2t}. \tag{25}$$

The operator $(1+\Delta)^s L$ is elliptic of order $m+2s$ so (25) and Gårding's inequality gives

$$(u, (1+\Delta)^s Lu + \lambda(1+\Delta)^s u)_0 \geq c_1\|u\|^2_{s+\frac{m}{2}} - c_2\|u\|^2_0 + \lambda\|u\|^2_s.$$

Since $\|u\|_s \geq \|u\|_0$ we can combine the two previous inequalities to get

$$\|u\|_t\|Lu+\lambda u\|_{t-m} \geq c_1\|u\|^2_t + (\lambda - c_2)\|u\|^2_0.$$

If $\lambda > c_2$ we can drop the second term and divide by $\|u\|_t$ to obtain (38).

We now prove the proposition for the case $t = \frac{m}{2} - s$ by the same sort of argument: We have

$$\|u\|_t \|Lu + \lambda u\|_{-s-\frac{m}{2}} = \|(1+\Delta)^{-s}u\|_{s+\frac{m}{2}} \|Lu + \lambda u\|_{-s-\frac{m}{2}}$$

$$\geq ((1+\Delta)^{-s}u, L(1+\Delta)^{s}(1+\Delta)^{-s}u + \lambda u)_0.$$

Now use the fact that $L(1+\Delta)^s$ is elliptic and Gårding's inequality to continue the above inequalities as

$$\geq c_1 \|(1+\Delta)^{-s}u\|^2_{s+\frac{m}{2}} - c_2\|(1+\Delta)^{-s}u\|_0^2 + \lambda \|u\|^2_{-s}$$

$$= c_1\|u\|_t^2 - c_2\|u\|_{-2s} + \lambda\|u\|^2_{-s} \geq c_1\|u\|_t^2$$

if $\lambda > c_2$. Again we may then divide by $\|u\|_t$ to get the result. QED

$$\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \tag{38}$$

The operator $L + \lambda I$ is a bounded operator from $\mathbf{H}_t$ to $\mathbf{H}_{t-m}$ (for any t). Suppose we fix t and choose λ so large that (38) holds. Then (38) says that $(L + \lambda I)$ is invertible on its image, and bounded there with a bound independent of $\lambda > \Lambda$, and this image is a closed subspace of $\mathbf{H}_{t-m}$.

Let us show that this image is all of $\mathbf{H}_{t-m}$ for λ large enough. Suppose not, which means that there is some $w \in \mathbf{H}_{t-m}$ with

$$(w, Lu + \lambda u)_{t-m} = 0$$

for all $u \in \mathbf{H}_t$. We can write this last equation as

$$((1 + \Delta)^{t-m} w, Lu + \lambda u)_0 = 0.$$

$$((1+\Delta)^{t-m}w, Lu+\lambda u)_0 = 0.$$

Integration by parts gives the adjoint differential operator L^* characterized by

$$(\phi, L\psi)_0 = (L^*\phi, \psi)_0$$

for all smooth functions ϕ and ψ, and by passing to the limit this holds for all elements of $\mathbf{H}_r$ for $r \geq m$. The operator L^* has the same leading term as L and hence is elliptic. So let us choose λ sufficiently large that (38) holds for L^* as well as for L. Now

$$0 = \left((1+\Delta)^{t-m}w, Lu+\lambda u\right)_0 = \left(L^*(1+\Delta)^{t-m}w + \lambda(1+\Delta)^{t-m}w, u\right)_0$$

for all $u \in \mathbf{H}_t$ which is dense in $\mathbf{H}_0$ so

$$L^*(1+\Delta)^{t-m}w + \lambda(1+\Delta)^{t-m}w = 0$$

and hence (by (38)) $(1+\Delta)^{t-m}w = 0$ so $w = 0$. We have proved

Proposition 4 *For every t and for λ large enough (depending on t) the operator $L+\lambda I$ maps $\mathbf{H}_t$ bijectively onto $\mathbf{H}_{t-m}$ and $(L+\lambda I)^{-1}$ is bounded independently of λ.*

Weyl's lemma.

As an immediate application we get the important

Theorem 6 *If u is a distribution and $Lu \in \mathbf{H}_s$ then $u \in \mathbf{H}_{s+m}$.*

Proof. Write $f = Lu$. By Schwartz' theorem, we know that $u \in \mathbf{H}_k$ for some k. So $f + \lambda u \in \mathbf{H}_{\min(k,s)}$ for any λ. Choosing λ large enough, we conclude that $u = (L + \lambda I)^{-1}(f + \lambda u) \in \mathbf{H}_{\min(k+m,s+m)}$. If $k + m < s + m$ we can repeat the argument to conclude that $u \in \mathbf{H}_{\min(k+2m,s+m)}$. we can keep going until we conclude that $u \in \mathbf{H}_{s+m}$. QED

Notice as an immediate corollary that any solution of the homogeneous equation $Lu = 0$ is C^∞.

Consequences of Weyl's lemma.

We now obtain a second important consequence of Proposition 4. Choose λ so large that the operators

$$(L+\lambda I)^{-1} \quad \text{and} \quad (L^*+\lambda I)^{-1}$$

exist as operators from $\mathbf{H}_0 \to \mathbf{H}_m$. Follow these operators with the injection $\iota_m : \mathbf{H}_m \to \mathbf{H}_0$ and set

$$M := \iota_m \circ (L+\lambda I)^{-1}, \qquad M^* := \iota_m \circ (L^*+\lambda I)^{-1}.$$

Since ι_m is compact (Rellich's lemma) and the composite of a compact operator with a bounded operator is compact, we conclude

Theorem 7 *The operators M and M^* are compact.*

Suppose that $L = L^*$. (This is usually expressed by saying that L is "formally self-adjoint". More on this terminology will come later.) This implies that $M = M^*$. In other words, M is a compact self adjoint operator, and we can apply Theorem 3 to conclude that eigenvectors of M form a basis of $R(M)$ and that the corresponding eigenvalues tend to zero. Prop 4 says that $R(M)$ is the same as $\iota_m(\mathbf{H}_m)$ which is dense in $\mathbf{H}_0 = L_2(\mathbf{T})$. We conclude that the eigenvectors of M form a basis of $L_2(\mathbf{T})$. If $Mu = ru$ then $u = (L + \lambda I)Mu = rLu + \lambda ru$ so u is an eigenvector of L with eigenvalue

$$\frac{1 - r\lambda}{r}.$$

We conclude that the eigenvectors of L are a basis of $\mathbf{H}_0$.

Main theorem.

$$\frac{1 - r\lambda}{r}.$$

We conclude that the eigenvectors of L are a basis of $\mathbf{H}_0$. We claim that only finitely many of these eigenvalues of L can be negative. Indeed, since we know that the eigenvalues r_n of M approach zero, the numerator in the above expression is positive, for large enough n, and hence if there were infinitely many negative eigenvalues μ_k, they would have to correspond to negative r_k and so these $\mu_k \to -\infty$. But taking $s_k = -\mu_k$ as the λ in (38) in Prop. 3 we conclude that $u = 0$, if $Lu = \mu_k u$ if k is large enough, contradicting the definition of an eigenvector. So all but a finite number of the r_n are positive, and these tend to zero. To summarize:

Theorem 8 *The eigenvectors of L are C^∞ functions which form a basis of $\mathbf{H}_0$. Only finitely many of the eigenvalues μ_k of L are negative and $\mu_n \to \infty$ as $n \to \infty$.*

Extensions.

Theorem 8 *The eigenvectors of L are C^∞ functions which form a basis of $\mathbf{H}_0$. Only finitely many of the eigenvalues μ_k of L are negative and $\mu_n \to \infty$ as $n \to \infty$.*

It is easy to extend the results obtained above for the torus in two directions. One is to consider functions defined in a **domain** = bounded open set $\mathcal{G}$ of $\mathbf{R}^n$ and the other is to consider functions defined on a compact manifold. In both cases a few elementary tricks allow us to reduce to the torus case. We sketch what is involved for the manifold case.

Let $E \to M$ be a vector bundle over a manifold. We assume that M is equipped with a density which we shall denote by $|dx|$ and that E is equipped with a positive definite (smoothly varying) scalar product, so that we can define the L_2 norm of a smooth section s of E of compact support:

$$\|s\|_0^2 := \int_M |s|^2(x)|dx|.$$

Suppose for the rest of this section that M is compact. Let $\{U_i\}$ be a finite cover of M by coordinate neighborhoods over which E has a given trivialization, and ρ_i a partition of unity subordinate to this cover. Let ϕ_i be a diffeomorphism or U_i with an open subset of $\mathbf{T}^n$ where n is the dimension of M. Then if s is a smooth section of E, we can think of $(\rho_i s) \circ \phi_i^{-1}$ as an $\mathbf{R}^m$ or $\mathbf{C}^m$ valued function on $\mathbf{T}^n$, and consider the sum of the $\|\cdot\|_k$ norms applied to each component. We shall continue to denote this sum by $\|\rho_i f \circ \phi_i^{-1}\|_k$ and then define

$$\|f\|_k := \sum_i \|\rho_i f \circ \phi_i^{-1}\|_k$$

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where the norms on the right are in the norms on the torus. These norms depend on the trivializations and on the partitions of unity. But any two norms are equivalent, and the $\| \ \|_0$ norm is equivalent to the "intrinsic" L_2 norm defined above. We define the Sobolev spaces $\mathbf{W}_k$ to be the completion of the space of smooth sections of E relative to the norm $\| \ \|_k$ for $k \geq 0$, and these spaces are well defined as topological vector spaces independently of the choices. Since Sobolev's lemma holds locally, it goes through unchanged. Similarly Rellich's lemma: if s_n is a sequence of elements of $\mathbf{W}_\ell$ which is bounded in the $\| \ \|_\ell$ norm for $\ell > k$, then each of the elements $\rho_i s_n \circ \phi_i^{-1}$ belong to $\mathbf{H}_\ell$ on the torus, and are bounded in the $\| \ \|_\ell$ norm, hence we can select a subsequence of $\rho_1 s_n \circ \phi_1^{-1}$ which converges in $\mathbf{H}_k$, then a sub-subsequence such that $\rho_i s_n \circ \phi_i^{-1}$ for $i = 1, 2$ converge etc. arriving at a subsequence of s_n which converges in $\mathbf{W}_k$.

Under changes of coordinates and trivializations the change in the coefficients are rather complicated, but the **symbol** of the differential operator

$$\sigma(L)(\xi) := \sum_{|p|=m} a_p(x)\xi^p \qquad \xi \in T^*M_x$$

is well defined.

If we put a Riemann metric on the manifold, we can talk about the length $|\xi|$ of any cotangent vector.

If L is a differential operator from E to itself (i.e. $F=E$) we shall call L **even elliptic** if m is even and there exists some constant C such that

$$\langle v, \sigma(L)(\xi)v\rangle \geq C|\xi|^m|v|^2$$

for all $x \in M$, $v \in E_x$, $\xi \in T^*M_x$ and $\langle\ ,\ \rangle$ denotes the scalar product on E_x. Gårding's inequality holds. Indeed, locally, this is just a restatement of the (vector valued version) of Gårding's inequality that we have already proved for the torus. But Stage 4 in the proof extends unchanged (other than the replacement of scalar valued functions by vector valued functions) to the more general case.