Math 212 Lecture 7

The Fourier transform.

Conventions, especially about 2π.

The space $\mathcal{S}$ consists of all functions on $\mathbf{R}$ which are infinitely differentiable and vanish at infinity rapidly with all their derivatives in the sense that

$$\|f\|_{m,n} := \sup\{|x^m f^{(n)}(x)|\} < \infty.$$

The $\|\cdot\|_{m,n}$ give a family of semi-norms on $\mathcal{S}$ making $\mathcal{S}$ into a Frechet space - that is, a vector space space whose topology is determined by a countable family of semi-norms. More about this later in the course. We use the measure

$$\frac{1}{\sqrt{2\pi}}dx$$

on $\mathbf{R}$ and so define the Fourier transform of an element of $\mathcal{S}$ by

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x)e^{-ix\xi}dx$$

and the convolution of two elements of $\mathcal{S}$ by

$$(f \star g)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x-t)g(t)dt.$$

Differentiation and multiplication by *x* .

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x) e^{-ix\xi} dx$$

The Fourier transform is well defined on $\mathcal{S}$ and

$$\left[\left(\frac{d}{dx} \right)^m ((-ix)^n f) \right]^{\wedge} = (i\xi)^m \left(\frac{d}{d\xi} \right)^n \hat{f},$$

as follows by differentiation under the integral sign and by integration by parts. This shows that the Fourier transform maps $\mathcal{S}$ to $\mathcal{S}$.

Convolution goes to multiplication.

$$\begin{aligned}
(f \star g)\hat{}(\xi) &= \frac{1}{2\pi} \int \int f(x-t)g(t)dx e^{-ix\xi} dx \\
&= \frac{1}{2\pi} \int \int f(u)g(t)e^{-i(u+t)\xi} du dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(u)e^{-iu\xi} du \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} g(t)e^{-it\xi} dt
\end{aligned}$$

so

$$(f \star g)\hat{} = \hat{f}\hat{g}.$$

Scaling.

For any $f \in \mathcal{S}$ and $a > 0$ define $S_a f$ by $(S_a)f(x) := f(ax)$. Then setting $u = ax$ so $dx = (1/a)du$ we have

$$\begin{aligned}
(S_a f)\hat{}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(ax) e^{-ix\xi} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (1/a) f(u) e^{-iu(\xi/a)} du
\end{aligned}$$

so

$$(S_a f)\hat{} = (1/a) S_{1/a} \hat{f}.$$

The Fourier transform of a Gaussian is a Gaussian.

The polar coordinate trick evaluates $\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-x^2/2}dx = 1.$

The integral

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-x^2/2-x\eta}dx$$

converges for all complex values of η, uniformly in any compact region. Hence it defines an analytic function of η that can be evaluated by taking η to be real and then using analytic continuation. For real η we complete the square and make a change of variables:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-x^2/2-x\eta}dx &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-(x+\eta)^2/2+\eta^2/2}dx \\ &= e^{\eta^2/2}\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-(x+\eta)^2/2}dx \\ &= e^{\eta^2/2}.\end{aligned}$$

Setting $\eta = i\xi$ gives

$$\hat{n} = n \quad \text{if } n(x) := e^{-x^2/2}.$$

Scaling the Gaussian.

If we set $a = \epsilon$ in our scaling equation and define

$$\rho_\epsilon := S_\epsilon n$$

so

$$\rho_\epsilon(x) = e^{-\epsilon^2 x^2/2},$$

then

$$(\rho_\epsilon)\check{}(x) = \frac{1}{\epsilon} e^{-x^2/2\epsilon^2}.$$

Notice that for any $g \in \mathcal{S}$ we have

$$\int_{\mathbf{R}} (1/a)(S_{1/a} g)(\xi) d\xi = \int_{\mathbf{R}} g(\xi) d\xi$$

so setting $a = \epsilon$ we conclude that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (\rho_\epsilon)\check{}(\xi) d\xi = 1$$

for all ϵ.

Let

$$\psi := \psi_1 := (\rho_1)\hat{}$$

and

$$\psi_\epsilon := (\rho_\epsilon)\hat{}.$$

Then

$$\psi_\epsilon(\eta) = \frac{1}{\epsilon}\psi\left(\frac{\eta}{\epsilon}\right)$$

so

$$(\psi_\epsilon \star g)(\xi) - g(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}[g(\xi-\eta) - g(\xi)]\frac{1}{\epsilon}\psi\left(\frac{\eta}{\epsilon}\right)d\eta =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}[g(\xi-\epsilon\zeta) - g(\xi)]\psi(\zeta)d\zeta.$$

Since $g \in \mathcal{S}$ it is uniformly continuous on $\mathbf{R}$, so that for any $\delta > 0$ we can find ϵ_0 so that the above integral is less than δ in absolute value for all $0 < \epsilon < \epsilon_0$. In short,

$$\|\psi_\epsilon \star g - g\|_\infty \to 0, \quad \text{as } \epsilon \to 0.$$

The multiplication formula.

This says that

$$\int_{\mathbf{R}} \hat{f}(x)g(x)dx = \int_{\mathbf{R}} f(x)\hat{g}(x)dx$$

for any $f, g \in \mathcal{S}$. Indeed the left hand side equals

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \int_{\mathbf{R}} f(y)e^{-ixy} dy g(x)dx.$$

We can write this integral as a double integral and then interchange the order of integration which gives the right hand side.

The inversion formula.

This says that for any $f \in \mathcal{S}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(\xi) e^{ix\xi} d\xi.$$

To prove this, we first observe that for any $h \in \mathcal{S}$ the Fourier transform of $x \mapsto e^{i\eta x} h(x)$ is just $\xi \mapsto \hat{h}(\xi - \eta)$ as follows directly from the definition.

Taking $g(x) = e^{itx} e^{-\epsilon^2 x^2/2}$ in the multiplication formula gives

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(t) e^{itx} e^{-\epsilon^2 t^2/2} dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(t) \psi_\epsilon(t - x) dt = (f \star \psi_\epsilon)(x).$$

We know that the right hand side approaches $f(x)$ as $\epsilon \to 0$. Also, $e^{-\epsilon^2 t^2/2} \to 1$ for each fixed t, and in fact uniformly on any bounded t interval. Furthermore, $0 < e^{-\epsilon^2 t^2/2} \leq 1$ for all t. So choosing the interval of integration large enough, we can take the left hand side as close as we like to $\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(x) e^{ixt} dt$ by then choosing ϵ sufficiently small. QED

Plancherel's theorem

Let

$$\tilde{f}(x) := \overline{f(-x)}.$$

Then the Fourier transform of $\tilde{f}$ is given by

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \overline{f(-x)} e^{-ix\xi} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \overline{f(u)} e^{iu\xi} du = \overline{\hat{f}(\xi)}$$

so

$$(\tilde{f})\hat{} = \overline{\hat{f}}.$$

Thus

$$(f \star \tilde{f})\hat{} = |\hat{f}|^2.$$

Plancherel's theorem , 2.

$$(f\star\tilde{f})\hat{} = |\hat{f}|^2.$$

The inversion formula applied to $f\star\tilde{f}$ and evaluated at 0 gives

$$(f\star\tilde{f})(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}|\hat{f}|^2dx.$$

The left hand side of this equation is

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}f(x)\tilde{f}(0-x)dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}|f(x)|^2dx.$$

Thus we have proved Plancherel's formula

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}|f(x)|^2dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}|\hat{f}(x)|^2dx.$$

Define $L_2(\mathbf{R})$ to be the completion of $\mathcal{S}$ with respect to the L_2 norm given by the left hand side of the above equation. Since $\mathcal{S}$ is dense in $L_2(\mathbf{R})$ we conclude that the Fourier transform extends to unitary isomorphism of $L_2(\mathbf{R})$ onto itself.

The Poisson summation formula.

This says that for any $g \in \mathcal{S}$ we have

$$\sum_k g(2\pi k) = \frac{1}{\sqrt{2\pi}} \sum_m \hat{g}(m).$$

To prove this let

$$h(x) := \sum_k g(x + 2\pi k)$$

so h is a smooth function, periodic of period 2π and

$$h(0) = \sum_k g(2\pi k).$$

We may expand h into a Fourier series

$$h(x) = \sum_m a_m e^{imx}$$

where

$$a_m = \frac{1}{2\pi} \int_0^{2\pi} h(x) e^{-imx} dx = \frac{1}{2\pi} \int_{\mathbf{R}} g(x) e^{-imx} dx = \frac{1}{\sqrt{2\pi}} \hat{g}(m).$$

$$h(x) := \sum_k g(x + 2\pi k) \qquad h(0) = \sum_k g(2\pi k).$$

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Setting $x = 0$ in the Fourier expansion

$$h(x) = \frac{1}{\sqrt{2\pi}}\sum \hat{g}(m)e^{imx}$$

gives

$$h(0) = \frac{1}{\sqrt{2\pi}}\sum_m \hat{g}(m).$$

so

$$\sum_k g(2\pi k) = \frac{1}{\sqrt{2\pi}}\sum_m \hat{g}(m).$$

The Shannon sampling theorem.

Let $f \in \mathcal{S}$ be such that its Fourier transform is supported in the interval $[-\pi, \pi]$. Then a knowledge of $f(n)$ for all $n \in \mathbf{Z}$ determines f. More explicitly,

$$f(t) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(n-t)}{n-t}. \tag{1}$$

Proof. Let g be the periodic function (of period 2π) which extends $\hat{f}$, the Fourier transform of f. So

$$g(\tau) = \hat{f}(\tau), \quad \tau \in [-\pi, \pi]$$

and is periodic.

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and is periodic.

Expand g into a Fourier series:

$$g = \sum_{n \in \mathbf{Z}} c_n e^{in\tau},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\tau) e^{-in\tau} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\tau) e^{-in\tau} d\tau,$$

or

$$c_n = \frac{1}{(2\pi)^{\frac{1}{2}}} f(-n).$$

But

$$f(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \hat{f}(\tau) e^{it\tau} d\tau = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\pi}^{\pi} g(\tau) e^{it\tau} d\tau =$$

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\pi}^{\pi} \sum \frac{1}{(2\pi)^{\frac{1}{2}}} f(-n) e^{i(n+t)\tau} d\tau.$$

$$f(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \hat{f}(\tau) e^{it\tau} d\tau = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\pi}^{\pi} g(\tau) e^{it\tau} d\tau =$$

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\pi}^{\pi} \sum \frac{1}{(2\pi)^{\frac{1}{2}}} f(-n) e^{i(n+t)\tau} d\tau.$$

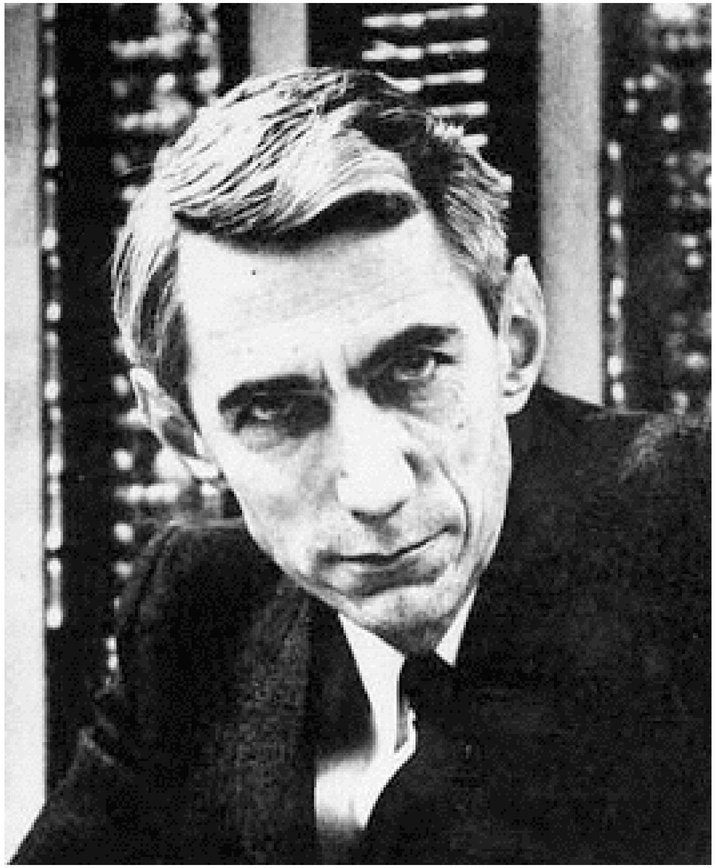
Replacing n by $-n$ in the sum, and interchanging summation and integration, which is legitimate since the $f(n)$ decrease very fast, this becomes

$$f(t) = \frac{1}{2\pi} \sum_n f(n) \int_{-\pi}^{\pi} e^{i(t-n)\tau} d\tau.$$

But

$$\int_{-\pi}^{\pi} e^{i(t-n)\tau} d\tau = \left. \frac{e^{i(t-n)\tau}}{i(t-n)} \right|_{-\pi}^{\pi} = \frac{e^{i(t-n)\pi} - e^{i(t-n)\pi}}{i(t-n)} = 2\frac{\sin \pi(n-t)}{n-t}. \text{ QED}$$

Claude E Shannon



Born: **30 April 1916 in Gaylord, Michigan, USA**
Died: **24 Feb 2001 in Medford, Massachusetts, USA**

It is useful to reformulate this via rescaling so that the interval $[-\pi, \pi]$ is replaced by an arbitrary interval symmetric about the origin: In the engineering literature the **frequency** λ is defined by

$$\xi = 2\pi\lambda.$$

Suppose we want to apply (1) to $g = S_a f$. We know that the Fourier transform of g is $(1/a)S_{1/a}\hat{f}$ and

$$\text{supp } S_{1/a}\hat{f} = a\text{supp } \hat{f}.$$

So if

$$\text{supp } \hat{f} \subset [-2\pi\lambda_c, 2\pi\lambda_c]$$

we want to choose a so that $a2\pi\lambda_c \leq \pi$ or

$$a \leq \frac{1}{2\lambda_c}. \tag{2}$$

Band width

For a in this range (1) says that

$$f(ax) = \frac{1}{\pi} \sum f(na) \frac{\sin \pi(x-n)}{x-n},$$

or setting $t = ax$,

$$f(t) = \sum_{n=-\infty}^{\infty} f(na) \frac{\sin(\frac{\pi}{a}(t-na))}{\frac{\pi}{a}(t-na)}. \qquad (3)$$

This holds in L_2 under the assumption that f satisfies supp $\hat{f} \subset [-2\pi\lambda_c, 2\pi\lambda_c]$. We say that f has **finite bandwidth** or is **bandlimited** with bandlimit λ_c. The critical value $a_c = 1/2\lambda_c$ is known as the **Nyquist sampling interval** and $(1/a) = 2\lambda_c$ is known as the **Nyquist sampling rate**. Thus the Shannon sampling theorem says that a band-limited signal can be recovered completely from a set of samples taken at a rate $\geq$ the Nyquist sampling rate.

The Heisenberg Uncertainty Principle.

Let $f \in \mathcal{S}(\mathbf{R})$ with

$$\int |f(x)|^2 dx = 1.$$

We can think of $x \mapsto |f(x)|^2$ as a probability density on the line. The mean of this probability density is

$$x_m := \int x|f(x)|^2 dx.$$

If we take the Fourier transform, then Plancherel says that

$$\int |\hat{f}(\xi)|^2 d\xi = 1$$

as well, so it defines a probability density with mean

$$\xi_m := \int \xi|\hat{f}(\xi)|^2 d\xi.$$

$$x_m := \int x|f(x)|^2 dx. \qquad \xi_m := \int \xi|\hat{f}(\xi)|^2 d\xi.$$

Suppose for the moment that these means both vanish. The **Heisenberg Uncertainty Principle** says that

$$\left(\int |xf(x)|^2 dx\right)\left(\int |\xi\hat{f}(\xi)|^2 d\xi\right) \geq \frac{1}{4}.$$

Proof. Write $-i\xi f(\xi)$ as the Fourier transform of f' and use Plancherel to write the second integral as $\int |f'(x)|^2 dx$. Then the Cauchy - Schwarz inequality says that the left hand side is $\geq$ the square of

$$\int |xf(x)f'(x)|dx \geq \left|\int \mathrm{Re}(xf(x)\overline{f'(x)})dx\right| =$$

$$\frac{1}{2}\left|\int x(f(x)\overline{f'(x)} + \overline{f(x)}f'(x)dx\right|$$

$$= \frac{1}{2}\left|\int x\frac{d}{dx}|f|^2 dx\right| = \frac{1}{2}\left|\int -|f|^2 dx\right| = \frac{1}{2}. \text{ QED}$$

If f has norm one but the mean of the probability density $|f|^2$ is not necessarily of zero (and similarly for for its Fourier transform) the Heisenberg uncertainty principle says that

$$\left(\int |(x - x_m)f(x)|^2 dx\right)\left(\int |(\xi - \xi_m)\hat{f}(\xi)|^2 d\xi\right) \geq \frac{1}{4}.$$

The general case is reduced to the special case by replacing $f(x)$ by

$$f(x + x_m)e^{i\xi_m x}.$$

Tempered distributions.

The space $\mathcal{S}$ was defined to be the collection of all smooth functions on $\mathbb{R}$ such that

$$\|f\|_{m,n} := \sup_x \{|x^m f^{(n)}(x)|\} < \infty.$$

The collection of these norms define a topology on $\mathcal{S}$ which is much finer that the L_2 topology: We declare that a sequence of functions $\{f_k\}$ approaches $g \in \mathcal{S}$ if and only if

$$\|f_k - g\|_{m,n} \to 0$$

for every m and n.

A linear function on $\mathcal{S}$ which is continuous with respect to this topology is called a **tempered distribution**.

Examples of tempered distributions.

A linear function on $\mathcal{S}$ which is continuous with respect to this topology is called a **tempered distribution**.

The space of tempered distributions is denoted by $\mathcal{S}'$. For example, every element $f \in \mathcal{S}$ defines a linear function on $\mathcal{S}$ by

$$\phi \mapsto \langle \phi, f \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(x) \overline{f(x)} dx.$$

But this last expression makes sense for any element $f \in L_2(\mathbb{R})$, or for any piecewise continuous function f which grows at infinity no faster than any polynomial. For example, if $f \equiv 1$, the linear function associated to f assigns to ϕ the value

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(x) dx.$$

This is clearly continuous with respect to the topology of $\mathcal{S}$ but this function of ϕ does not make sense for a general element ϕ of $L_2(\mathbb{R})$.

Examples of tempered distributions.

Another example of an element of $\mathcal{S}'$ is the Dirac δ-function which assigns to $\phi \in \mathcal{S}$ its value at 0. This is an element of $\mathcal{S}'$ but makes no sense when evaluated on a general element of $L_2(\mathbb{R})$.

The Fourier transform of tempered distributions.

If $f \in \mathcal{S}$, then the Plancherel formula formula implies that its Fourier transform $\mathcal{F}(f) = \hat{f}$ satisfies

$$(\phi, f) = (\mathcal{F}(\phi), \mathcal{F}(f)).$$

But we can now use this equation to *define* the Fourier transform of an arbitrary element of $\mathcal{S}'$: If $\ell \in \mathcal{S}'$ we define $\mathcal{F}(\ell)$ to be the linear function

$$\mathcal{F}(\ell)(\psi) := \ell(\mathcal{F}^{-1}(\psi)).$$

Examples of Fourier transform of tempered distributions.

- If ℓ corresponds to the function $f \equiv 1$, then

$$\mathcal{F}(\ell)(\psi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}^{-1}\psi)(\xi) d\xi = \mathcal{F}\left(\mathcal{F}^{-1}\psi\right)(0) = \psi(0).$$

So the Fourier transform of the function which is identically one is the Dirac δ-function.

- If δ denotes the Dirac δ-function, then

$$(\mathcal{F}(\delta)(\psi) = \delta(\mathcal{F}^{-1}(\psi)) = \left((\mathcal{F}^{-1}(\psi)\right)(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x) dx.$$

So the Fourier transform of the Dirac δ function is the function which is identically one.

- In fact, this last example follows from the preceding one: If $m = \mathcal{F}(\ell)$ then

$$(\mathcal{F}(m)(\phi) = m(\mathcal{F}^{-1}(\phi)) = \ell(\mathcal{F}^{-1}(\mathcal{F}^{-1}(\phi)).$$

  But

$$\mathcal{F}^{-2}(\phi)(x) = \phi(-x).$$

  So if $m = \mathcal{F}(\ell)$ then $\mathcal{F}(m) = \breve{\ell}$ where

$$\breve{\ell}(\phi) := \ell(\phi(-\bullet)).$$

The Fourier transform of the function x: This assigns to every $\psi \in \mathcal{S}$ the value

$$\frac{1}{\sqrt{2\pi}} \int \psi(\xi) e^{ix\xi} x d\xi dx = \frac{1}{\sqrt{2\pi}} \int \psi(\xi) \frac{1}{i} \frac{d}{d\xi} \left(e^{ix\xi} \right) d\xi dx =$$

$$i \frac{1}{\sqrt{2\pi}} \int \frac{d\psi(\xi)}{dx} e^{ix\xi} d\xi dx = i \left(\mathcal{F} \left((\mathcal{F}^{-1} \left(\frac{d\psi(\xi)}{dx} \right) \right) \right) (0) = i\delta \left(\frac{d\psi(\xi)}{dx} \right).$$

Now for an element of $\mathcal{S}$ we have

$$\int \frac{d\phi}{dx} \cdot \overline{f} dx = -\frac{1}{\sqrt{2\pi}} \int \phi \overline{\frac{df}{dx}} dx.$$

So we define the derivative of an $\ell \in \mathcal{S}'$ by

$$\frac{d\ell}{dx}(\phi) = \ell \left(-\frac{d\phi}{dx} \right).$$

So the Fourier transform of x is $-i \frac{d\delta}{dx}$.