Math 212 Lecture 10.

The Hausdorff dimension of fractals.

Review: Hausdorff measure.

Let X be a metric space. Recall that if A is any subset of X, the **diameter** of A is defined as

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x,y).$$

Take $\mathcal{C}$ to be the collection of all subsets of X, and for any positive real number s define

$$\ell_s(A) = \operatorname{diam}(A)^s$$

(with $0^s = 0$). Take $\mathcal{C}$ to consist of all subsets of X. The method II outer measure is called the s-**dimensional Hausdorff outer measure**, and its restriction to the associated σ-field of (Caratheodory) measurable sets is called the s-**dimensional Hausdorff measure**.

Review: Hausdorff dimension.

Theorem 6 *Let $F \subset X$ be a Borel set. Let $0 < s < t$. Then*

$$\mathcal{H}_s(F) < \infty \;\Rightarrow \mathcal{H}_t(F) = 0$$

and

$$\mathcal{H}_t(F) > 0 \;\Rightarrow \mathcal{H}_s(F) = \infty.$$

This last theorem implies that for any Borel set F, there is a unique value s_0 (which might be 0 or ∞) such that $\mathcal{H}_t(F) = \infty$ for all $t < s_0$ and $\mathcal{H}_s(F) = 0$ for all for all $s > s_0$. This value is called the **Hausdorff dimension** of F. It is one of many competing (and non-equivalent) definitions of dimension. Notice that it is a metric invariant, and in fact is the same for two spaces different by homeomorphism with Lipschitz inverse.

Contracting ratio lists.

A finite collection of real numbers

$$(r_1, \ldots, r_n)$$

is called a **contracting ratio list** if

$$0 < r_i < 1 \quad \forall\, i = 1, \ldots, n.$$

Proposition 12 *Let $(r_1, \ldots, r_n)$ be a contracting ratio list. There exists a unique non-negative real number s such that*

$$\sum_{i=1}^{n} r_i^s = 1. \tag{20}$$

The number s is 0 if and only if $n = 1$.

Proof. If $n = 1$ then $s = 0$ works and is clearly the only solution. If $n > 1$, define the function f on $[0, \infty)$ by

$$f(t) := \sum_{i=1}^{n} r_i^t.$$

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We have

$$f(0) = n \quad \text{and} \quad \lim_{t\to\infty} f(t) = 0 < 1.$$

Since f is continuous, there is some postive solution to (20). To show that this solution is unique, it is enough to show that f is monotone decreasing. This follows from the fact that its derivative is

$$\sum_{i=1}^{n} r_i^t \log r_i < 0.$$

QED

Similarity dimension.

$$\sum_{i=1}^{n} r_i^s = 1. \tag{20}$$

Definition 12.1 *The number s in (20) is called the* **similarity dimension** *of the ratio list $(r_1, \ldots, r_n)$.*

Iterated function systems.

A map $f : X \to Y$ between two metric spaces is called a **similarity** with similarity ratio r if

$$d_Y(f(x_1), f(x_2)) = r d_X(x_1, x_2) \;\; \forall \, x_1, \; x_2 \in X.$$

(Recall that a map is called **Lipschitz** with Lipschitz constant r if we only had an inequality, $\leq$, instead of an equality in the above.

Let X be a complete metric space, and let $(r_1, \ldots, r_n)$ be a contracting ratio list. A collection

$$(f_1, \ldots, f_n), \quad f_i : \; X \to X$$

is called an **iterated function system** which **realizes** the contracting ratio list if

$$f_i : X \to X, \quad i = 1, \ldots, n$$

is a similarity with ratio r_i. We also say that $(f_1, \ldots, f_n)$ is a **realization** of the ratio list $(r_1, \ldots, r_n)$.

The goal of today's lecture.

It is a consequence of *Hutchinson's theorem*, see below, that

Proposition 13 *If* $(f_1, \ldots, f_n)$ *is a realization of the contracting ratio list* $(r_1, \ldots, r_n)$ *on a complete metric space,* X*, then there exists a unique non-empty compact subset* $K \subset X$ *such that*

$$K = f_1(K) \cup \cdots \cup f_n(K).$$

In fact, Hutchinson's theorem asserts the corresponding result where the f_i are merely assumed to be Lipschitz maps with Lipschitz constants $(r_1, \ldots, r_n)$.

The set K is sometimes called the fractal associated with the realization $(f_1, \ldots, f_n)$ of the contracting ratio list $(r_1, \ldots, r_n)$. The facts we want to establish are: First,

$$\dim(K) \leq s \tag{21}$$

where dim denotes Hausdorff dimension, and s is the similarity dimension of $(r_1, \ldots, r_n)$. In general, we can only assert an inequality here, for the the set K does not fix $(r_1, \ldots, r_n)$ or its realization. For example, we can repeat some of the r_i and the corresponding f_i. This will give us a longer list, and hence a larger s, but will not change K. But we can demand a rather strong form of non-redundancy known as **Moran's condition**: There exists an open set U such that

$$U \supset f_i(U) \; \forall \; i \quad \text{and} \quad f_i(U) \cap f_j(U) = \emptyset \;\; \forall \; i \neq j. \tag{22}$$

Then

Theorem 7 *If $(f_1, \ldots, f_n)$ is a realization of $(r_1, \ldots, r_n)$ on $\mathbf{R}^d$ and if Moran's condition holds then*

$$\dim K = s.$$

$$\dim(K) \leq s \tag{21}$$

The method of proof of (21) will be to construct a “model” complete metric space E with a realization $(g_1, \ldots, g_n)$ of $(r_1, \ldots, r_n)$ on it, which is “universal” in the sense that

- E is itself the fractal associated to $(g_1, \ldots, g_n)$.
- The Hausdorff dimension of E is s.
- If $(f_1, \ldots, f_n)$ is a realization of $(r_1, \ldots, r_n)$ on a complete metric space X then there exists a unique continuous map

  $$h : E \to X$$

  such that

  $$h \circ g_i = f_i \circ h. \tag{23}$$

- The image $h(E)$ of h is K.
- The map h is Lipschitz.

This is clearly enough to prove (21). A little more work will then prove Moran's theorem.

Constructing the string model.

Let $(r_1, \ldots, r_n)$ be a contracting ratio list, and let $\mathcal{A}$ denote the alphabet consisting of the letters $\{1, \ldots, n\}$. Let E denote the space of one sided infinite strings of letters from the alphabet $\mathcal{A}$. If α denotes a finite string (word) of letters from $\mathcal{A}$, we let w_α denote the product over all i occurring in α of the r_i. Thus

$$w_\emptyset = 1$$

where $\emptyset$ is the empty string, and, inductively,

$$w_{\alpha e} = w_\alpha \cdot w_e, \quad e \in \mathcal{A}.$$

If $x \neq y$ are two elements of E, they will have a longest common initial string α, and we then define

$$d(x, y) := w_\alpha.$$

This makes E into a complete ultrametic space. Define the maps $g_i : E \to E$ by

$$g_i(x) = ix.$$

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$$g_i(x) = ix.$$

That is, g_i shifts the infinite string one unit to the right and inserts the letter i in the initial position. In terms of our metric, clearly $(g_1, \ldots, g_n)$ is a realization of $(r_1, \ldots, r_n)$ and the space E itself is the corresponding fractal set.

We let $[\alpha]$ denote the set of all strings beginning with α, i.e. whose first word (of length equal to the length of α) is α. The diameter of this set is w_α.

The Hausdorff dimension of E is s.

We begin with a lemma:

Lemma 4 *Let $A \subset E$ have positive diameter. Then there exists a word α such that $A \subset [\alpha]$ and*

$$\operatorname{diam}(A) = \operatorname{diam}[\alpha] = w_\alpha.$$

Proof. Since A has at least two elements, there will be a γ which is a prefix of one and not the other. So there will be an integer n (possibly zero) which is the length of the longest common prefix of all elements of A. Then every element of A will begin with this common prefix α which thus satisfies the coniditions of the lemma. QED

$$\sum_{i=1}^{n} r_i^s = 1. \tag{20}$$

Lemma 4 *Let $A \subset E$ have positive diameter. Then there exists a word α such that $A \subset [\alpha]$ and*

$$\mathrm{diam}(A) = \mathrm{diam}[\alpha] = w_\alpha.$$

The lemma implies that in computing the Hausdorff measure or dimension, we need only consider covers by sets of the form $[\alpha]$. Now if we choose s to be the solution of (20), then

$$(\mathrm{diam}[\alpha])^s = \sum_{i=1}^{n} (\mathrm{diam}[\alpha i])^s = (\mathrm{diam}[\alpha])^s \sum_{i=1}^{n} r_i^s.$$

This means that the method II outer measure assosicated to the function $A \mapsto (\mathrm{diam}\, A)^s$ coinsides with the method I outer measure and assigns to each set $[\alpha]$ the measure w_α^s. In particular the measure of E is one, and so the Hausdorff dimension of E is s.

The universality of the string model.

Let $(f_1, \ldots, f_n)$ a realization of $(r_1, \ldots, r_n)$ on a complete metric space X. Choose a point $a \in X$ and define $h_0 : E \to X$ by

$$h_0(z) :\equiv a.$$

Inductively define the maps h_p by defining h_{p+1} on each of the open sets $[\{i\}]$ by

$$h_{p+1}(iz) := f_i(h_p(z)).$$

The sequence of maps $\{h_p\}$ is Cauchy in the uniform norm. Indeed, if $y \in [\{i\}]$ so $y = g_i(z)$ for some $z \in E$ then

$$d_X\left(h_{p+1}(y), h_p(y)\right) = d_X\left(f_i(h_p(z)), f_i(h_{p-1}(z))\right)$$

$$= r_i d_X\left(h_p(z), h_{p-1}(z)\right)).$$

$$d_X\left(h_{p+1}(y), h_p(y)\right) = \quad r_i d_X\left(h_p(z), h_{p-1}(z)\right)).$$

So if we let $c := \max_i(r_i)$ so that $0 < c < 1$, we have

$$\sup_{y \in E} d_X\left(h_{p+1}(y), h_p(y)\right) \leq c \sup_{x \in E} d_X\left(h_p(x), h_p(x)\right)$$

for $p \geq 1$ and hence

$$\sup_{y \in E} d_X\left(h_{p+1}(y), h_p(y)\right) < Cc^p$$

for a suitable constant C. This shows that the h_p converge uniformly to a limit h which satisfies

$$h \circ g_i = f_i \circ h.$$

$$h \circ g_i = f_i \circ h.$$

Now

$$h_{k+1}(E) = \bigcup_i f_i\left(h_k(E)\right),$$

and the proof of Hutchinson's theorem given below - using the contraction fixed point theorem for compact sets under the Hausdorff metric - shows that the sequence of sets $h_k(E)$ converges to the fractal K.

Since the image of h is K which is compact, the image of $[\alpha]$ is $f_\alpha(K)$ where we are using the obvious notation $f_{ij} = f_i \circ f_j$, $f_{ijk} = f_i \circ f_j \circ f_k$ etc. The set $f_\alpha(K)$ has diameter $w_\alpha \cdot \mathrm{diam}(K)$. Thus h is Lipschitz with Lipschitz constant $\mathrm{diam}(K)$.

The uniqueness of the map h follows from the above sort of argument.

The Hausdorff metric.

Let X be a complete metric space. Let $\mathcal{H}(X)$ denote the space of non-empty compact subsets of X. For any $A \in \mathcal{H}(X)$ and any positive number ϵ, let

$$A_\epsilon = \{x \in X | d(x, y) \leq \epsilon, \text{for some } y \in A\}.$$

We call A_ϵ the ϵ-*collar* of A. Recall that we defined

$$d(x, A) = \inf_{y \in A} d(x, y)$$

to be the distance from any $x \in X$ to A, then we can write the definition of the ϵ-collar as

$$A_\epsilon = \{x | d(x, A) \leq \epsilon\}.$$

Notice that the infimum in the definition of $d(x, A)$ is actually achieved, that is, there is some point $y \in A$ such that

$$d(x, A) = d(x, y).$$

This is because A is compact.

Notice that the infimum in the definition of $d(x, A)$ is actually achieved, that is, there is some point $y \in A$ such that

$$d(x, A) = d(x, y).$$

This is because A is compact. For a pair of non-empty compact sets, A and B, define

$$d(A, B) = \max_{x \in A} d(x, B).$$

So

$$d(A, B) \leq \epsilon \;\; iff \;\; A \subset B_\epsilon.$$

Notice that this condition is not symmetric in A and B. So Hausdorff introduced

$$\begin{aligned} h(A, B) &= \max\{d(A, B), d(B, A)\} && (25) \\ &= \inf\{\epsilon \mid A \subset B_\epsilon \;\text{ and }\; B \subset A_\epsilon\}. && (26) \end{aligned}$$

as a distance on $\mathcal{H}(X)$. He proved

$$
\begin{aligned}
h(A,B) &= \max\{d(A,B), d(B,A)\} \quad (25)\\
&= \inf\{\epsilon \mid A \subset B_\epsilon \text{ and } B \subset A_\epsilon\}. \quad (26)
\end{aligned}
$$

Proposition 14 *The function h on $\mathcal{H}(X) \times \mathcal{H}(X)$ satsifies the axioms for a metric and makes $\mathcal{H}(X)$ into a complete metric space. Furthermore, if*

$$A, B, C, D \in \mathcal{H}(X)$$

then

$$h(A \cup B, C \cup D) \leq \max\{h(A,C), h(B,D)\}. \quad (27)$$

Proof. We begin with (27). If ϵ is such that $A \subset C_\epsilon$ and $B \subset D_\epsilon$ then clearly $A \cup B \subset C_\epsilon \cup D_\epsilon = (C \cup D)_\epsilon$. Repeating this argument with the roles of A, C and B, D interchanged proves (27).

$$h(A,B) = \max\{d(A,B), d(B,A)\} \quad (25)$$
$$= \inf\{\epsilon \mid A \subset B_\epsilon \ \text{ and } \ B \subset A_\epsilon\}. \quad (26)$$

We prove that h is a metric: h is symmetric, by definition. Also, $h(A,A) = 0$, and if $h(A,B) = 0$, then every point of A is within zero distance of B,and hence must belong to B since B is compact, so $A \subset B$ and similarly $B \subset A$. So $h(A,B) = 0$ implies that $A = B$.

We must prove the triangle inequality. For this it is enough to prove that

$$d(A,B) \leq d(A,C) + d(C,B),$$

because interchanging the role of A and B gives the desired result.

Now for any $a \in A$ we have

$$\begin{aligned} d(a,B) &= \min_{b\in B} d(a,b) \\ &\leq \min_{b\in B} \left(d(a,c) + d(c,b)\ \forall c \in C\right. \\ &= d(a,c) + \min_{b\in B} d(c,b)\ \forall c \in C \\ &= d(a,c) + d(c,B)\ \forall c \in C \\ &\leq d(a,c) + d(C,B)\ \forall c \in C. \end{aligned}$$

The second term in the last expression does not depend on c, so minimizing over c gives

$$d(a,B) \leq d(a,C) + d(C,B).$$

Maximizing over a on the right gives

$$d(a,B) \leq d(A,C) + d(C,B).$$

Maximizing on the left gives the desired

$$d(A,B) \leq d(A,C) + d(C,A).$$

We sketch the proof of completeness. Let A_n be a sequence of compact non-empty subsets of X which is Cauchy in the Hausdorff metric. Define the set A to be the set of all $x \in X$ with the property that there exists a sequence of points $x_n \in A_n$ with $x_n \to x$. It is straighforward to prove that A is compact and non-empty and is the limit of the A_n in the Hausdorff metric.

Contractions.

Suppose that $\kappa : X \to X$ is a contraction. Then κ defines a transformation on the space of subsets of X (which we continue to denote by κ):

$$\kappa(A) = \{\kappa x | x \in A\}.$$

Since κ is continuous, it carries $\mathcal{H}(X)$ into itself. Let c be the Lipschitz constant of κ. Then

$$\begin{aligned} d(\kappa(A), \kappa(B)) &= \max_{a \in A}[\min_{b \in B} d(\kappa(a), \kappa(b))] \\ &\leq \max_{a \in A}[\min_{b \in B} cd(a, b)] \\ &= cd(A, B). \end{aligned}$$

Similarly, $d(\kappa(B), \kappa(A)) \leq c\, d(B, A)$ and hence

$$h(\kappa(A), \kappa(B)) \leq c\, h(A, B). \tag{28}$$

In other words, a contraction on X induces a contraction on $\mathcal{H}(X)$.

Families of contractions.

$$h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\}. \tag{27}$$

Proposition 15 *Let $T_1, \ldots, T_n$ be a collection of contractions on $\mathcal{H}(X)$ with Lipschitz constants $c_1, \ldots, c_n$, and let $c = \max c_i$. Define the transformation T on $\mathcal{H}(X)$ by*

$$T(A) = T_1(A) \cup T_2(A) \cup \cdots \cup T_n(A).$$

Then T is a contraction with Lipschitz constant c.

Proof. By induction, it is enough to prove this for the case $n = 2$. By (27)

$$\begin{aligned} h(T(A), T(B)) &= h(T_1(A) \cup T_2(A), T_1(B) \cup T_2(B)) \\ &\leq \max\{h(T_1(A), h(T_1(B)), h(T_2(A), T_2(B))\} \\ &\leq \max\{c_1 h(A, B), c_2 h(A, B)\} \\ &= h(A, B) \max\{c_1, c_2\} = c \cdot h(A, B) \end{aligned}$$

Hutchinson's theorem.

Putting the previous facts together we get Hutchinson's theorem;

Theorem 8 *Let $T_1, \ldots, T_n$ be contractions on a complete metric space and let c be the maximum of their Lipschitz contants. Define the Hutchinoson operator T on $\mathcal{H}(X)$ by*

$$T(A) := T_1(A) \cup \cdots \cup T_n(A).$$

Then T is a contraction with Lipschtz constant c.

John Hutchinson

Fractals.

We can now apply the contraction fixed point theorem to T to conclude that

Proposition 13 *If $(f_1, \ldots, f_n)$ is a realization of the contracting ratio list $(r_1, \ldots, r_n)$ on a complete metric space, X, then there exists a unique non-empty compact subset $K \subset X$ such that*

$$K = f_1(K) \cup \cdots \cup f_n(K).$$

We have proved that the Hausdorff dimension of *K* is at most *s* where s is the similarity dimension of the contracting ratio list $(r_1, \ldots, r_n)$.

The classical Cantor set.

Take $X = [0,1]$, the unit interval. Take

$$T_1 : x \mapsto \frac{x}{3}, \quad T_2 : x \mapsto \frac{x}{3} + \frac{2}{3}.$$

These are both contractions, so by Hutchinson's theorem there exists a unique closed fixed set C. This is the Cantor set.

To relate it to Cantor's original construction, let us go back to the proof of the contraction fixed point theorem applied to T acting on $\mathcal{H}(X)$. It says that if we start with any non-empty compact subset A_0 and keep applying T to it, i.e. set $A_n = T^n A_0$ then $A_n \to C$ in the Hausdorff metric, h. Suppose we take the interval I itself as our A_0. Then

$$A_1 = T(I) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1].$$

in other words, applying the Hutchinson operator T to the interval $[0,1]$ has the effect of deleting the "middle third" open interval $(\frac{1}{3}, \frac{2}{3})$.

Applying T once more gives

$$A_2 = T^2[0,1] = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1].$$

In other words, A_2 is obtained from A_1 by deleting the middle thirds of each of the two intervals of A_1 and so on. This was Cantor's original construction. Since $A_{n+1} \subset A_n$ for this choice of initial set, the Hausdorff limit coincides with the intersection.

But of course Hutchinson's theorem (and the proof of the contractions fixed point theorem) says that we can start with *any* non-empty closed set as our initial "seed" and then keep applying T. For example, suppose we start with the one point set $B_0 = \{0\}$. Then $B_1 = TB_0$ is the two point set

$$B_1 = \{0, \frac{2}{3}\},$$

B_2 consists of the four point set

$$B_2 = \{0, \frac{2}{9}, \frac{2}{3}, \frac{8}{9}\}$$

and so on. We then must take the Hausdorff limit of this increasing collection of sets.

To describe the limiting set c from this point of view, it is useful to use triadic expansions of points in $[0, 1]$. Thus

$$\begin{aligned} 0 &= .0000000\cdots \\ 2/3 &= .2000000\cdots \\ 2/9 &= .0200000\cdots \\ 8/9 &= .2200000\cdots \end{aligned}$$

and so on. Thus the set B_n will consist of points whose triadic expansion has only zeros or twos in the first n positions followed by a string of all zeros. Thus a point will lie in C (be the limit of such points) if and only if it has a triadic expansion consisting entirely of zeros or twos. This includes the possibility of an infinite string of all twos at the tail of the expansion. for example, the point 1 which belongs to the Cantor set has a triadic expansion $1 = .222222\cdots$.

Similarly the point $\frac{2}{3}$ has the triadic expansion $\frac{2}{3} = .0222222\cdots$ and so is in the limit of the sets B_n. But a point such as $.101\cdots$ is not in the limit of the B_n and hence not in C. This description of C is also due to Cantor. Notice that for any point a with triadic expansion $a = .a_1a_2a_2\cdots$

$$T_1a = .0a_1a_2a_3\cdots, \quad \text{while} \quad T_2a = .2a_1a_2a_3\cdots.$$

Thus if all the entries in the expansion of a are either zero or two, this will also be true for T_1a and T_2a. This shows that the C (given by this second Cantor description) satisfies $TC \subset C$. On the other hand,

$$T_1(.a_2a_3\cdots) = .0a_2a_3\cdots, \qquad T_2(.a_2a_3\cdots) = .2a_2a_3\cdots$$

which shows that $.a_1a_2a_3\cdots$ is in the image of T_1 if $a_1 = 0$ or in the image of T_2 if $a_1 = 2$. This shows that $TC = C$. Since C (according to Cantor's second description) is closed, the uniqueness part of the fixed point theorem guarantees that the second description coincides with the first.

The statement that $TC = C$ implies that C is "self-similar".

The Sierpinski Gasket

Consider the three affine transformations of the plane:

$$T_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}, \quad T_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$T_3 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The fixed point of the Hutchinson operator for this choice of T_1, T_2, T_3 is called the Sierpinski gasket, S. If we take our initial set A_0 to be the right triangle with vertices at

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then each of the $T_i A_0$ is a similar right triangle whose linear dimensions are one-half as large, and which shares one common vertex with the original triangle.

In other words,

$$A_1 = TA_0$$

is obtained from our original triangle be deleting the interior of the (reversed) right triangle whose vertices are the midpoints of our orignal triangle. Just as in the case of the Cantor set, successive applications of T to this choice of original set amounts to successive deletions of the "middle" and the Hausdorff limit is the intersection of all of them: $S = \bigcap A_i$.

We can also start with the one element set

$$B_0 \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

Using a binary expansion for the x and y coordinates, application of T to B_0 gives the three element set

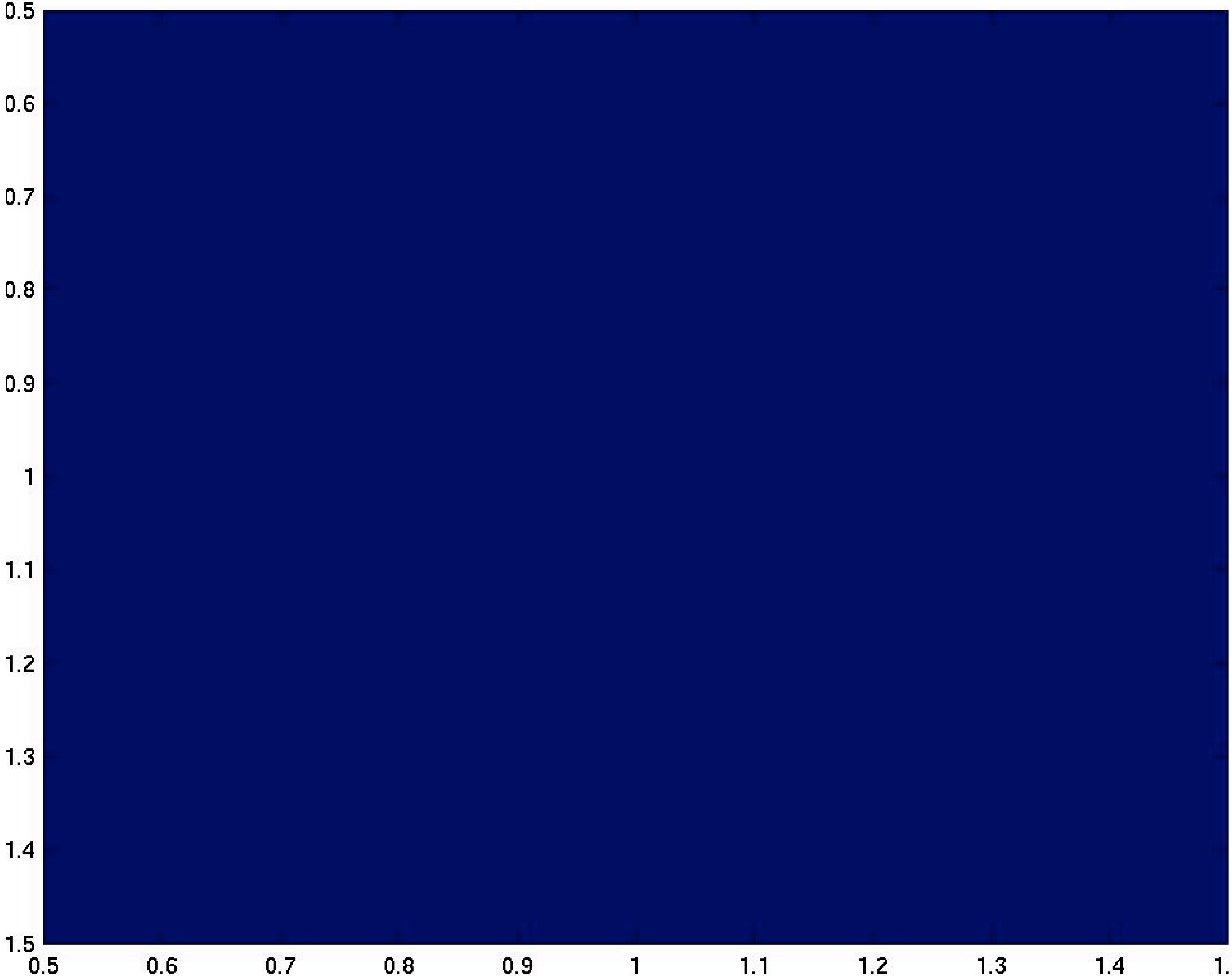
$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} .1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ .1 \end{pmatrix} \right\}.$$

The set $B_2 = TB_1$ will contain nine points, whose binary expansions are obtained from the above three by shifting the x and y exapnsions one unit to the right and either inserting a 0 before both expansions (the effect of T_1), insert a 1 before the expansion of x and a zero before the y or vice versa. Proceding in this fashion, we see that B_n consists of 3^n points which have all 0 in the binary expansion of the x and y coordinates, past the n-th position, and which are further constrained by the condition that at no earler point do we have both $x_i = 1$ and $y_i = 1$. Passing to the limit shows that S consists of all points for which we can find (possible inifinite) binary expansions of the x and y coordinates so that $x_i = 1 = y_i$ never occurs. (For example $x = \frac{1}{2}, y = \frac{1}{2}$ belongs to S because we can write $x = .10000\cdots, y = .011111\ldots$). Again, from this (second) description of S in terms of binary expansions it is clear that $TS = S$.

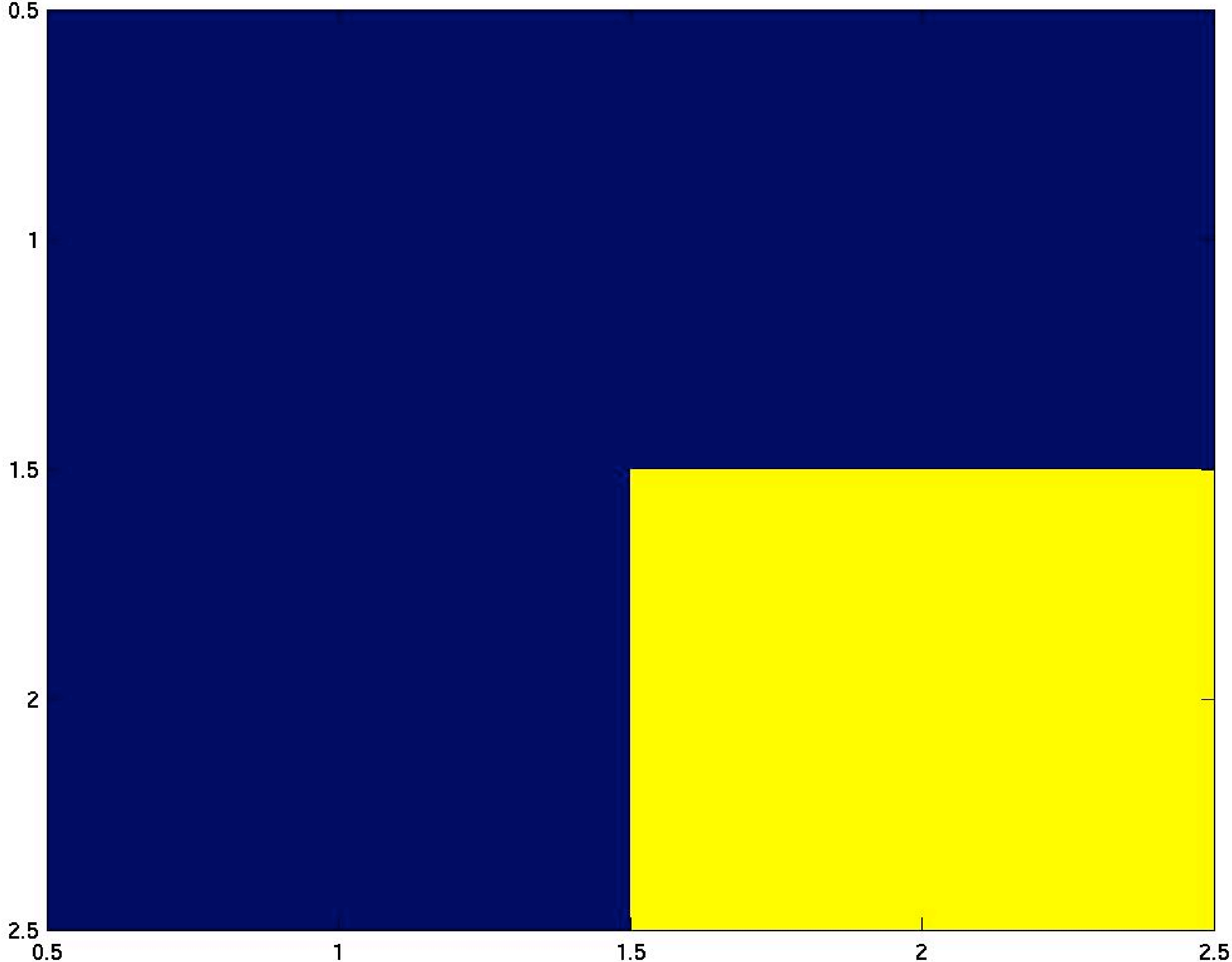
A one line code for creating the Sierpinski gasket.

The following is a matlab m file for doing the first seven approximations to the Sierpinksi gasket as a "movie". Notice that that iterative scheme is encoded in the single line "J=[J J;J zeros(2^i,2^i)];". The other instructions are for the graphics, etc. This shows the power of Hutchinson's theorem and also raises the philosophical question as to the notion of "simplicity".

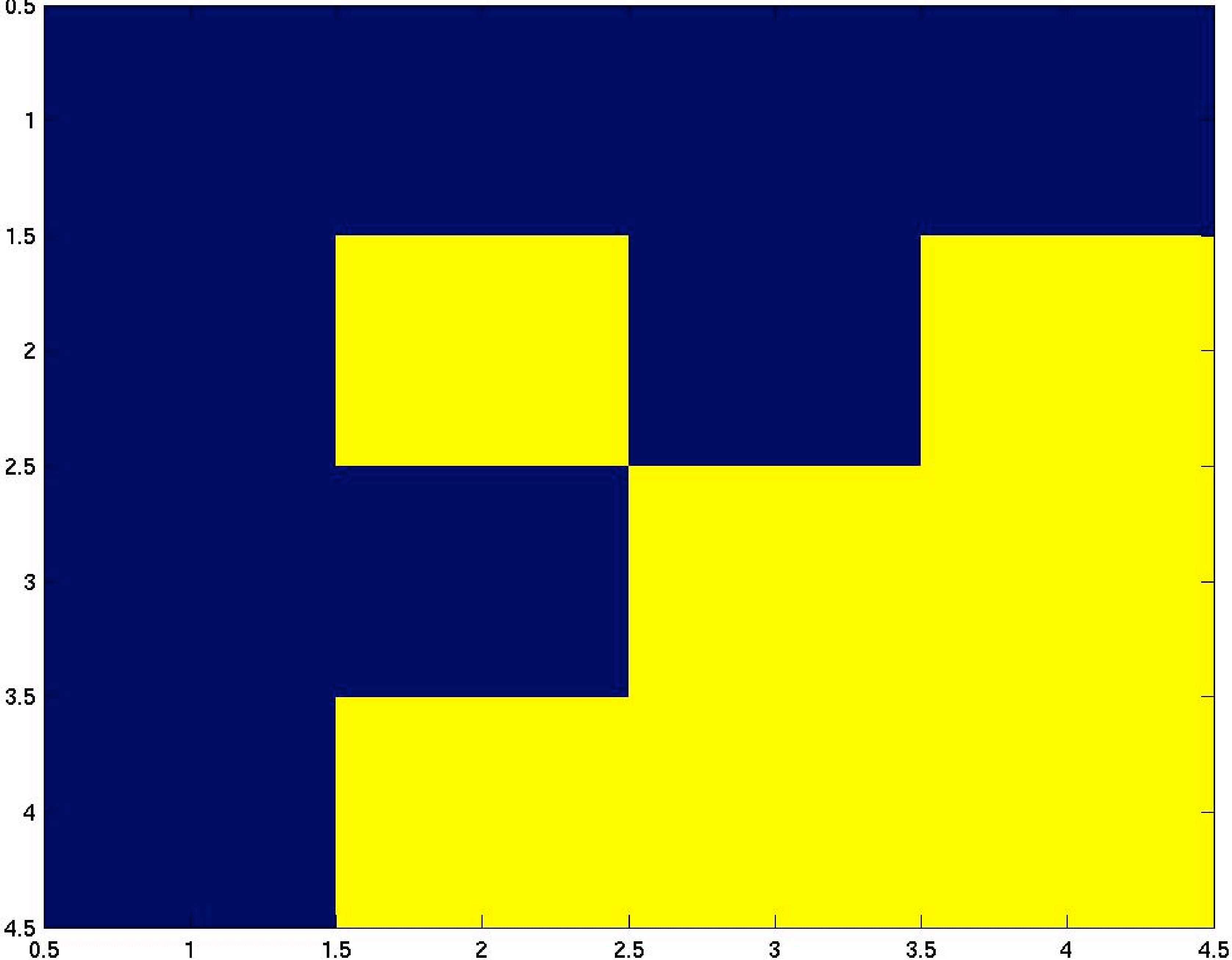
```
J=[10];
image(J);colormap(colorcube(17))
pause(3)
for i=0:6
        J=[J J;J zeros(2^i,2^i)];
        image(J);
colormap(colorcube(17));
        pause(3)
end
```



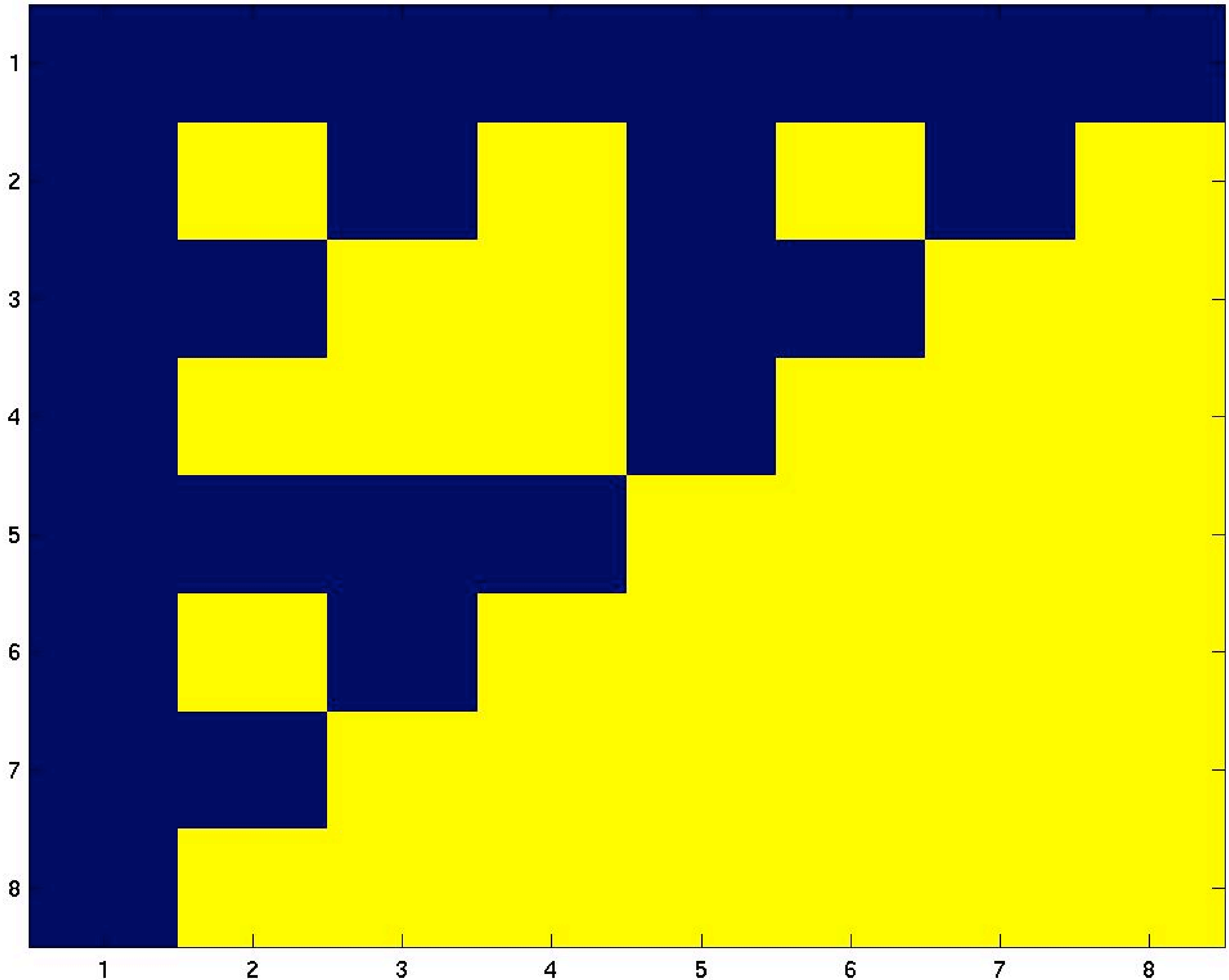
Stage 1.



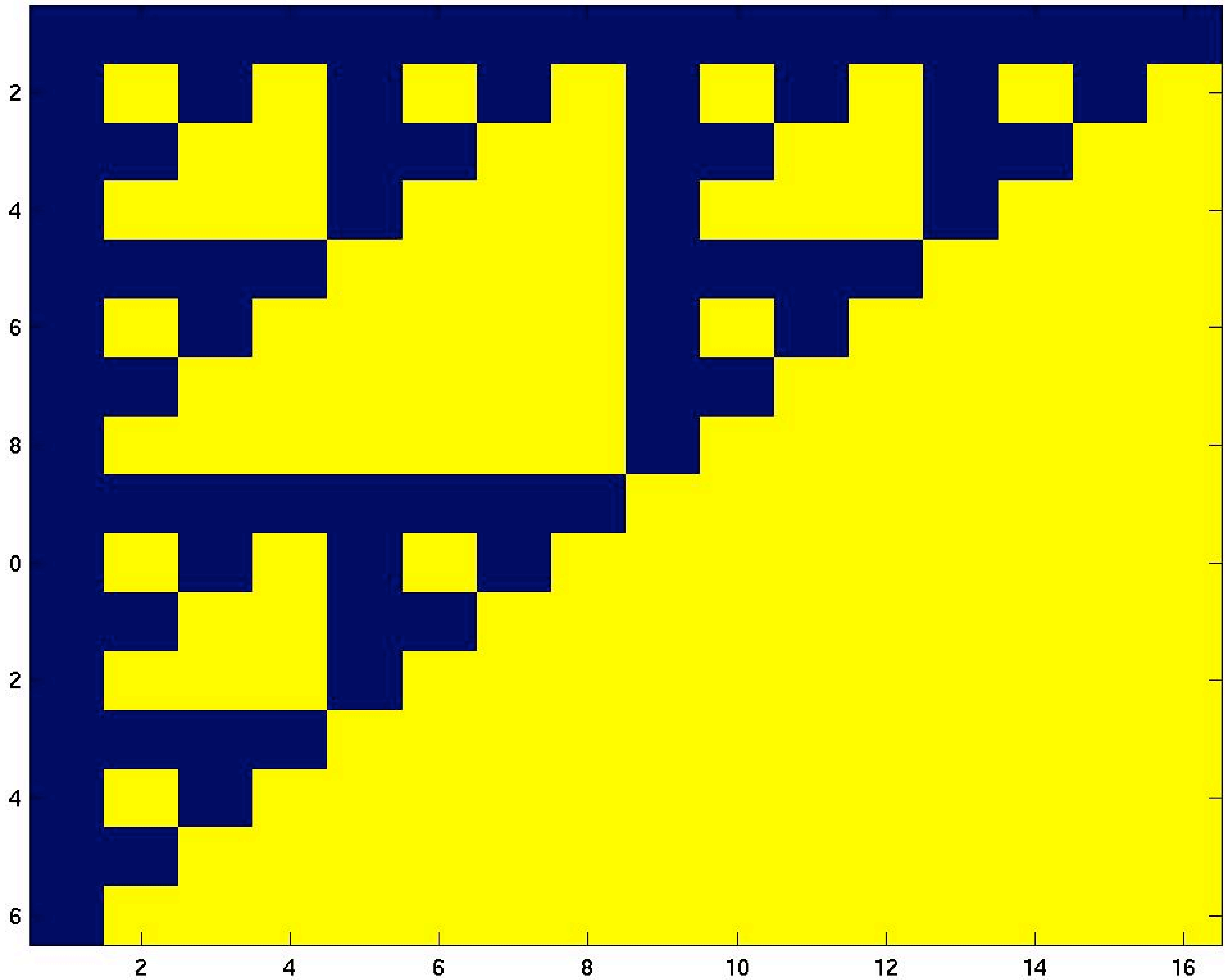
Stage 2.



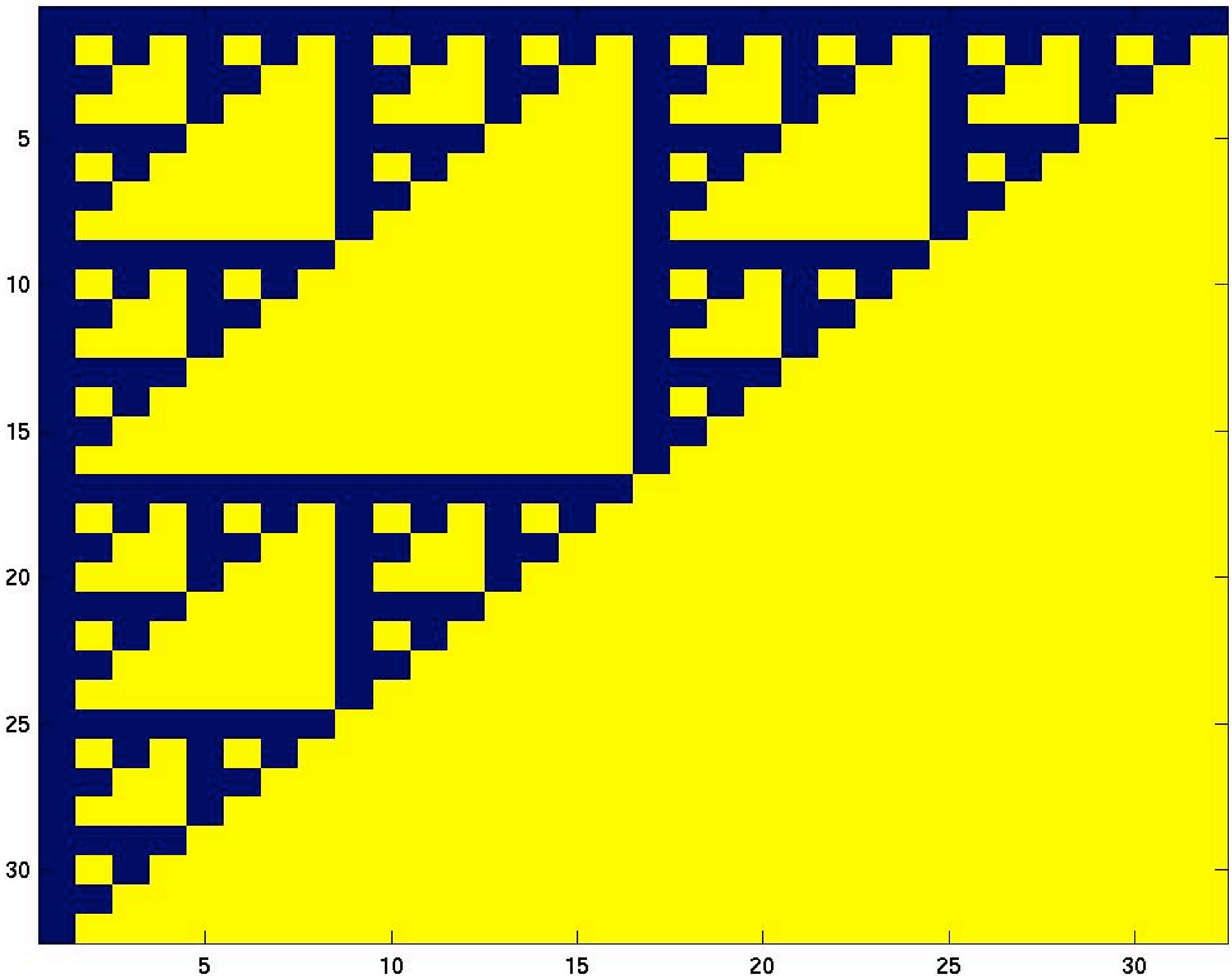
Stage 3.



Stage 4.



Stage 5.



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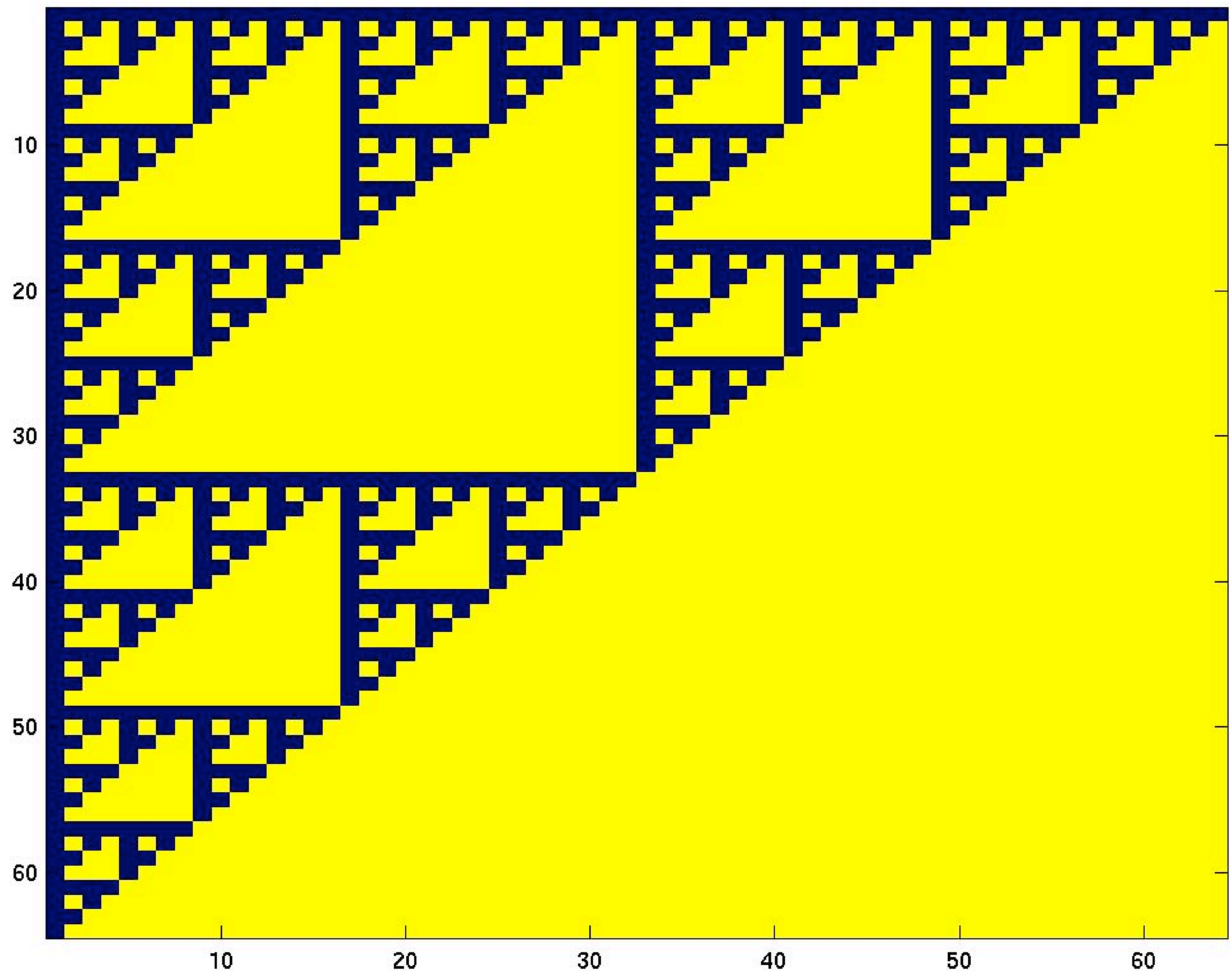

Stage 7.

Moran's theorem.

Moran's condition: There exists an open set O such that

$$O \supset f_i(O) \; \forall \; i \quad \text{and} \quad f_i(O) \cap f_j(O) = \emptyset \quad \forall \; i \neq j. \tag{22}$$

Theorem 7 *If* $(f_1, \ldots, f_n)$ *is a realization of* $(r_1, \ldots, r_n)$ *on* $\mathbf{R}^d$ *and if Moran's condition holds then*

$$\dim K = s.$$

Let $\mathfrak{m}$ denote the measure on the string model E that we constructed above, so that $\mathfrak{m}(E) = 1$ and more generally $\mathfrak{m}([\alpha]) = w_\alpha^s$. Then we will have proved that the Hausdorff dimension of K is $\geq s$, and hence $= s$ if we can prove that there exists a constant b such that for every Borel set $B \subset K$

$$\mathfrak{m}(h^{-1}(B)) \leq b \cdot \operatorname{diam}(B)^s, \tag{29}$$

where $h : E \to K$ is the map we constructed above from the string model to K.

If A is any set such that $f[A] \subset A$, then clearly $f^p[A] \subset A$ by induction. If A is non-empty and closed, then for any $a \in A$, and any $x \in E$, the limit of the $f_\gamma(a)$ belongs to K as γ ranges over the first words of size p of x, and so belongs to K and also to A. Since these points consititute all of K, we see that

$$K \subset A$$

and hence

$$f_\beta(K) \subset f_\beta(A) \tag{28}$$

for any word β.

Now suppose that Moran's open set condition is satisfied, and let us write

$$O_\alpha := f_\alpha(O).$$

$$O_\alpha := f_\alpha(O). \qquad f_\beta(K) \subset f_\beta(A) \tag{28}$$

Then

$$O_\alpha \cap O_\beta = \emptyset$$

if α is not a prefix of β or β is not a prefix of α. Furthermore,

$$\overline{f_\beta(O)} = f_\beta(\overline{O})$$

so we can use the symbol

$$\overline{O}_\beta$$

unambiguously to denote these two equal sets. By virtue of (28) we have

$$K_\beta \subset \overline{O}_\beta$$

where we use K_β to denote $f_\beta(K)$. Suppose that α is not a prefix of β or vice versa. Then $K_\beta \cap O\alpha = \emptyset$ since $\overline{O}_\beta \cap O_\alpha = \emptyset$.

Let us introduce the following notation: For any (finite) non-empty string α, let α^- denote the string (of cardinality one less) obtained by removing the last letter in α.

Lemma 5 *There exists an integer N such that for any subset $B \subset K$ the set Q_B of all finite strings α such that*

$$\overline{O_\alpha} \cap B \neq \emptyset$$

and

$$\operatorname{diam} O_\alpha < \operatorname{diam} B \leq \operatorname{diam} O_{\alpha^-}$$

has at most N elements.

Proof. Let

$$D := \operatorname{diam} O.$$

The map f_α is a similarity with similarity ratio $\operatorname{diam}[\alpha]$ so

$$\operatorname{diam} O_\alpha = D \cdot \operatorname{diam}[\alpha].$$

Let $r := \min_i r_i$. Then if $\alpha \in Q_B$ we have

$$\operatorname{diam} O_\alpha = D \cdot \operatorname{diam}[\alpha] \geq D \cdot r \operatorname{diam}[\alpha^-] = r \operatorname{diam} O_{\alpha^-} \geq r \operatorname{diam} B.$$

$$\operatorname{diam} O_\alpha = D \cdot \operatorname{diam}[\alpha] \geq D \cdot r \operatorname{diam}[\alpha^-] = r \operatorname{diam} O_{\alpha^-} \geq r \operatorname{diam} B.$$

Let V denote the volume of O relative to the d-dimensional Hausdorff measure of $\mathbb{R}^d$, i.e., up to a constant factor the Lebesgue measure. Let V_α denote the volume of O_α so that $V_\alpha = w_\alpha^d V = V \cdot (\operatorname{diam} O_\alpha)/\operatorname{diam} O)^d$. From the preceding displayed equation it follows that

$$V_\alpha \geq \frac{V r^d}{D^d} (\operatorname{diam} B)^d.$$

If $x \in B$, then every $y \in O_\alpha$ is within a distance $\operatorname{diam} B + \operatorname{diam} O_\alpha \leq 2 \operatorname{diam} B$ of x. So if m denotes the number of elements in Q_B, we have m disjoint sets with volume at least $\frac{V r^d}{D^d} (\operatorname{diam} B)^d$ all within a ball of radius $2 \cdot \operatorname{diam} B$. We have normalized our volume so that the unit ball has volume one, and hence the ball of radius $2 \cdot \operatorname{diam} B$ has volume $2^d (\operatorname{diam} B)^d$. Hence

$$m \cdot \frac{V r^d}{D^d} (\operatorname{diam} B)^d \leq 2^d (\operatorname{diam} B)^d$$

or

$$m \leq \frac{2^d D^d}{V r^d}.$$

So any integer greater that the right hand side of this inequality (which is independent of B) will do. $\square$

Now we turn to the proof of (29) which will then complete the proof of Moran's theorem. Let B be a Borel subset of K. Then

$$B \subset \bigcup_{\alpha \in Q_B} \overline{O_\alpha}$$

so

$$h^{-1}(B) \subset \bigcup_{\alpha \in Q_B} [\alpha].$$

Now

$$([\alpha]) = (\operatorname{diam}[\alpha])^s = \left(\frac{1}{D} \operatorname{diam}(O_\alpha)\right)^s \leq \frac{1}{D^s} (\operatorname{diam} B)^s$$

and so

$$\mathfrak{m}(h^{-1}(B)) \leq \sum_{\alpha \in Q_B} \mathfrak{m}(\alpha) \leq N \cdot \frac{1}{D^s} (\operatorname{diam} B)^s$$

and hence we may take

$$b = N \cdot \frac{1}{D^s} (\operatorname{diam} B)^s$$

and then (29) will hold.