Math 212 Lecture 11.

The Lebesgue integral.

Review: sigma fields.

Let X be a set.

(Usually X will be a topological space or even a metric space). A collection $\mathcal{F}$ of subsets of X is called a σ field if:

- $X \in \mathcal{F}$,
- If $E \in \mathcal{F}$ then $E^c = X \setminus E \in \mathcal{F}$, and
- If $\{E_n\}$ is a sequence of elements in $\mathcal{F}$ then $\bigcup_n E_n \in \mathcal{F}$,

The intersection of any family of σ-fields is again a σ-field, and hence given any collection $\mathcal{C}$ of subsets of X, there is a smallest σ-field $\mathcal{F}$ which contains it. Then $\mathcal{F}$ is called the σ-field **generated** by $\mathcal{C}$.

If X is a metric space, the σ-field generated by the collection of open sets is called the **Borel** σ-field, usually denoted by $\mathcal{B}$ or $\mathcal{B}(X)$ and a set belonging to $\mathcal{B}$ is called a **Borel set**.

Review: measures on sigma fields.

Given a σ-field $\mathcal{F}$ a (non-negative) **measure** is a function

$$m : \mathcal{F} \to [0, \infty]$$

such that

- $m(\emptyset) = 0$ and
- **Countable additivity:** If F_n is a disjoint collection of sets in $\mathcal{F}$ then
$$m\left(\bigcup_n F_n\right) = \sum_n m(F_n).$$

In the countable additivity condition it is understood that both sides might be infinite.

A key tool.

Let $A_n \in \mathcal{F}$ with $A_n \subset A_{n+1}$ and

$$A := \bigcup A_n.$$

We describe this situation by writing

$$A_n \nearrow A.$$

If m is a measure on $\mathcal{F}$ then

$$\lim_{n\to\infty} m(A_n) = m(A).$$

This is the key fact that we will be using over and over again.

In what follows, $(X, \mathcal{F}, m)$ is a space with a σ-field of sets, and m a measure on $\mathcal{F}$. The purpose of this chapter is to develop the theory of the Lebesgue integral for functions defined on X. The theory starts with **simple** functions, that is functions which take on only finitely many non-zero values, say $\{a_1, \ldots, a_n\}$ and where

$$A_i := f^{-1}(a_i) \in \mathcal{F}.$$

In other words, we start with functions of the form

$$\phi(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i} \qquad A_i \in \mathcal{F}. \tag{1}$$

Then, for any $E \in \mathcal{F}$ we would like to define the integral of a simple function ϕ over E as

$$\int_E \phi dm = \sum_{i=1}^{n} a_i m(A_i \cap E) \tag{2}$$

and extend this definition by some sort of limiting process to a broader class of functions.

The function values.

$$\int_E \phi dm = \sum_{i=1}^{n} a_i m(A_i \cap E) \qquad (2)$$

I haven't yet specified what the range of the functions should be. Certainly, even to get started, we have to allow our functions to take values in a vector space over $\mathbf{R}$, in order that the expression on the right of (2) make sense. In fact, I will eventually allow f to take values in a Banach space. However the theory is a bit simpler for real valued functions, where the linear order of the reals makes some arguments easier. Of course it would then be no problem to pass to any finite dimensional space over the reals. But we will on occasion need integrals in infinite dimensional Banach spaces, and that will require a little reworking of the theory.

Real valued measurable functions.

Recall that if $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are spaces with σ-fields, then

$$f : X \to Y$$

is called measurable if

$$f^{-1}(E) \in \mathcal{F} \quad \forall\ E \in \mathcal{G}. \tag{3}$$

Notice that the collection of subsets of Y for which (3) holds is a σ-field, and hence if it holds for some collection $\mathcal{C}$, it holds for the σ-field generated by $\mathcal{C}$. For the next few sections we will take $Y = \mathbf{R}$ and $\mathcal{G} = \mathcal{B}$, the Borel field. Since the collection of open intervals on the line generate the Borel field, a real valued function $f : X \to \mathbf{R}$ is measurable if and only if

$$f^{-1}(I) \in \mathcal{F} \quad \text{for all open intervals } \ I.$$

Equally well, it is enough to check this for intervals of the form $(-\infty, a)$ for all real numbers a.

Compositions of measurable functions.

Proposition 1 *If* $F : \mathbf{R}^2 \to \mathbf{R}$ *is a continuous function and* f, g *are two measurable real valued functions on* X, *then* $F(f,g)$ *is measurable.*

Proof. The set $F^{-1}(-\infty, a)$ is an open subset of the plane, and hence can be written as the countable union of products of open intervals $I \times J$. So if we set $h = F(f,g)$ then $h^{-1}((-\infty, a))$ is the countable union of the sets $f^{-1}(I) \cap g^{-1}(J)$ and hence belongs to $\mathcal{F}$. QED

From this elementary proposition we conclude that if f and g are measurable real valued functions then

From this elementary proposition we conclude that if f and g are measurable real valued functions then

- $f + g$ is measurable (since $(x, y) \mapsto x + y$ is continuous),
- fg is measurable (since $(x, y) \mapsto xy$ is continuous), hence
- $f\mathbf{1}_A$ is measurable for any $A \in \mathcal{F}$ hence
- f^+ is measurable since $f^{-1}([0, \infty]) \in \mathcal{F}$ and similarly for f^- so
- $|f|$ is measurable and so is $|f - g|$. Hence
- $f \wedge g$ and $f \vee g$ are measurable

and so on.

The integral of a non-negative function.

We are going to allow for the possibility that an integral might be infinite. We adopt the convention that

$$0 \cdot \infty = 0.$$

Recall that ϕ is **simple** if ϕ takes on a finite number of distinct non-negative values, $a_1, \ldots, a_n$, and that each of the sets

$$A_i = \phi^{-1}(a_i)$$

is measurable. These sets partition X:

$$X = A_1 \cup \cdots \cup A_n.$$

Of course since the values are distinct,

$$A_i \cap A_j = \emptyset \quad \text{for} \quad i \neq j.$$

$$\phi(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i} \qquad A_i \in \mathcal{F}. \quad (1) \qquad\qquad \int_E \phi dm = \sum_{i=1}^{n} a_i m(A_i \cap E) \quad (2)$$

With this definition, a simple function can be written as in (1) and this expression is unique. So we may take (2) as the definition of the integral of a simple function. We now extend the definition to an arbitrary ($[0,\infty]$ valued) function f by

$$\int_E f dm := \sup I(E, f) \tag{4}$$

where

$$I(E,f) = \left\{ \int_E \phi dm : 0 \leq \phi \leq f,\ \phi \text{ simple} \right\}. \tag{5}$$

Notice that if $A := f^{-1}(\infty)$ has positive measure, then the simple functions $n\mathbf{1}_A$ are all $\leq f$ and so $\int_X f dm = \infty$.

Proposition 2 *For simple functions, the definition (4) coincides with definition (2).*

Proof. Since ϕ is $\leq$ itself, the right hand side of (2) belongs to $I(E, \phi)$ and hence is $\leq \int_E \phi dm$ as given by (5). We must show the reverse inequality: Suppose that $\psi = \sum b_j \mathbf{1}_{B_j} \leq \phi$. We can write the right hand side of (2) as

$$\sum b_j m(E \cap B_j) = \sum_{i,j} b_j m(E \cap A_i \cap B_j)$$

since $E \cap B_j$ is the disjoint union of the sets $E \cap A_i \cap B_j$ because the A_i partition X, and m is additive on disjoint finite (even countable) unions. On each of the sets $A_i \cap B_j$ we must have $b_j \leq a_i$. Hence

$$\sum_{i,j} b_j m(E \cap A_i \cap B_j) \leq \sum_{i,j} a_i m(E \cap A_i \cap B_j) = \sum a_i m(E \cap A_i)$$

since the B_j partition X. QED

In the course of the proof of the above proposition we have also established

$$\psi \leq \phi \ \text{ for simple functions implies } \int \psi dm \leq \int \phi dm. \qquad (6)$$

$$\int_E \phi dm = \sum_{i=1}^{n} a_i m(A_i \cap E) \qquad (2)$$

Suppose that E and F are disjoint measurable sets. Then

$$m(A_i \cap (E \cup F)) = m(A_i \cap E) + m(A_i \cap F)$$

so each term on the right of (2) breaks up into a sum of two terms and we conclude that

If ϕ is simple and $E \cap F = \emptyset$, then $\int_{E \cup F} \phi dm = \int_E \phi dm + \int_F \phi dn$ (7)

Also, it is immediate from (2) that if $a \geq 0$ then

$$\text{If } \phi \text{ is simple then } \int_E a\phi dm = a \int_E \phi dm. \qquad (8)$$

It is now immediate that these results extend to all non-negative measurable functions. We list the results and then prove them. In what follows f and g are non-negative measurable functions, $a \geq 0$ is a real number and E and F are measurable sets:

$$
\begin{aligned}
f \le g \quad &\Rightarrow \quad \int_E f dm \le \int_E g dm. && (9)\\
\int_E f dm \quad &= \quad \int_X \mathbf{1}_E f dm && (10)\\
E \subset F \quad &\Rightarrow \quad \int_E f dm \le \int_F f dm. && (11)\\
\int_E a f dm \quad &= \quad a \int_E f dm. && (12)\\
m(E) = 0 \quad &\Rightarrow \quad \int_E f dm = 0. && (13)\\
E \cap F = \emptyset \quad &\Rightarrow \quad \int_{E \cup F} f dm = \int_E f dm + \int_F f dm. && (14)\\
f = 0 \ \text{ a.e.} \quad &\Leftrightarrow \quad \int_X f dm = 0. && (15)\\
f \le g \ \text{ a.e.} \quad &\Rightarrow \quad \int_X f dm \le \int_X g dm. && (16)
\end{aligned}
$$

Proofs:

$$f \le g \quad \Rightarrow \quad \int_E f\,dm \le \int_E g\,dm. \tag{9}$$

$$\int_E f\,dm \quad = \quad \int_X \mathbf{1}_E f\,dm \tag{10}$$

$$E \subset F \quad \Rightarrow \quad \int_E f\,dm \le \int_F f\,dm. \tag{11}$$

$$\int_E af\,dm \quad = \quad a\int_E f\,dm. \tag{12}$$

(9): $I(E,f) \subset I(E,g)$.

(10): If ϕ is a simple function with $\phi \le f$, then multiplying ϕ by $\mathbf{1}_E$ gives a function which is still $\le f$ and is still a simple function. The set $I(E,f)$ is unchanged by considering only simple functions of the form $\mathbf{1}_E\phi$ and these constitute all simple functions $\le \mathbf{1}_E f$.

(11): We have $\mathbf{1}_E f \le \mathbf{1}_F f$ and we can apply (9) and (10).

(12): $I(E,af) = aI(E,f)$.

Proofs, continued.

$$\int_E afdm = a\int_E fdm. \quad (12)$$

$$m(E) = 0 \Rightarrow \int_E fdm = 0. \quad (13)$$

(12): $I(E, af) = aI(E, f)$.

(13): In the definition (2) all the terms on the right vanish since $m(E \cap A_i) = 0$. So $I(E, f)$ consists of the single element 0.

$$E \cap F = \emptyset \quad \Rightarrow \quad \int_{E \cup F} f dm = \int_E f dm + \int_F f dm. \qquad (14)$$

(14): This is true for simple functions, so $I(E \cup F, f) = I(E, f) + I(F, f)$ meaning that every element of $I(E \cup F, f)$ is a sum of an element of $I(E, f)$ and an element of $I(F, f)$. Thus the sup on the left is $\leq$ the sum of the sups on the right, proving that the left hand side of (14) is $\leq$ its right hand side. To prove the reverse inequality, choose a simple function $\phi \leq \mathbf{1}_E f$ and a simple function $\psi \leq \mathbf{1}_F f$. Then $\phi + \psi \leq \mathbf{1}_{E \cup F} f$ since $E \cap F = \emptyset$. So $\phi + \psi$ is a simple function $\leq f$ and hence

$$\int_E \phi dm + \int_F \psi dm \leq \int_{E \cup F} f dm.$$

If we now maximize the two summands separately we get

$$\int_E f dm + \int_F f dm \leq \int_{E \cup F} f dm$$

which is what we want.

$$f = 0 \text{ a.e.} \quad \Leftrightarrow \quad \int_X f\,dm = 0. \tag{15}$$

(15): If $f = 0$ almost everywhere, and $\phi \leq f$ then $\phi = 0$ a.e. since $\phi \geq 0$. This means that all sets which enter into the right hand side of (2) with $a_i \neq 0$ have measure zero, so the right hand side vanishes. So $I(X, f)$ consists of the single element 0. This proves $\Rightarrow$ in (15). We wish to prove the reverse implication. Let $A = \{x | f(x) > 0\}$. We wish to show that $m(A) = 0$. Now

$$A = \bigcup A_n \quad \text{where} \quad A_n := \{x | f(x) > \frac{1}{n}\}.$$

The sets A_n are increasing, so we know that $m(A) = \lim_{n\to\infty} m(A_n)$. So it is enough to prove that $m(A_n) = 0$ for all n. But

$$\frac{1}{n}\mathbf{1}_{A_n} \leq f$$

and is a simple function. So

$$\frac{1}{n}\int_X \mathbf{1}_{A_n} dm = \frac{1}{n} m(A_n) \leq \int_X f\,dm = 0$$

implying that $m(A_n) = 0$.

$$f \leq g \ \text{ a.e.} \quad \Rightarrow \quad \int_X f dm \leq \int_X g dm. \tag{16}$$

(16): Let $E = \{x | f(x) \leq g(x)\}$. Then E is measurable and E^c is of measure zero. By definition, $\mathbf{1}_E f \leq \mathbf{1}_E g$ everywhere, hence by (11)

$$\int_X \mathbf{1}_E f dm \leq \int_X \mathbf{1}_E g dm.$$

But

$$\int_X \mathbf{1}_E f dm + \int_X \mathbf{1}_{E^c} f dm = \int_E f dm + \int_{E^c} f dm = \int_X f dm$$

where we have used (14) and (13). Similarly for g. QED

Fatou's lemma.

Theorem 1 *If* $\{f_n\}$ *is a sequence of non-negative functions, then*

$$\lim_{n\to\infty} \inf_{k\geq n} \int f_k dm \geq \int \left(\lim_{n\to\infty} \inf_{k\geq n} f_k \right) dm. \tag{17}$$

Recall that the limit inferior of a sequence of numbers $\{a_n\}$ is defined as follows: Set

$$b_n := \inf_{k\geq n} a_k$$

so that the sequence $\{b_n\}$ is non-decreasing, and hence has a limit (possibly infinite) which is defined as the lim inf. For a sequence of functions, lim inf f_n is obtained by taking lim inf $f_n(x)$ for every x.

Strict inequality can occur.

Consider the sequence of simple functions $\{\mathbf{1}_{[n,n+1]}\}$. At each point x the lim inf is 0, in fact $\mathbf{1}_{[n,n+1]}(x)$ becomes and stays 0 as soon as $n > x$. Thus the right hand side of (17) is zero. The numbers which enter into the left hand side are all 1, so the left hand side is 1.

Similarly, if we take $f_n = n\mathbf{1}_{(0,1/n]}$, the left hand side is 1 and the right hand side is 0. So without further assumptions, we generally expect to get strict inequality in Fatou's lemma.

Proof of Fatou's lemma.

Set

$$g_n := \inf_{k \geq n} f_k$$

so that

$$g_n \leq g_{n+1}$$

and set

$$f := \lim_{n \to \infty} \inf_{k \geq n} f_n = \lim_{n \to \infty} g_n.$$

Let

$$\phi \leq f$$

be a simple function. We must show that

$$\int \phi dm \leq \lim_{n \to \infty} \inf_{k \geq n} \int f_k dm. \tag{18}$$

To show:

$$\int \phi dm \leq \lim_{n\to\infty} \inf_{k\geq n} \int f_k dm. \tag{18}$$

There are two cases to consider:

a) $m(\{x : \phi(x) > 0\}) = \infty$. In this case $\int \phi dm = \infty$ and hence $\int f dm = \infty$ since $\phi \leq f$. We must show that $\liminf \int f_n dm = \infty$. Let

$$D := \{x : \phi(x) > 0\} \quad \text{so } m(D) = \infty.$$

Choose some positive number $b <$ all the positive values taken by ϕ. This is possible since there are only finitely many such values.

Let

$$D_n := \{x | g_n(x) > b\}.$$

The $D_n \nearrow D$ since $b < \phi(x) \leq \lim_{n\to\infty} g_n(x)$ at each point of D. Hence $m(D_n) \to m(D) = \infty$.

$$D_n := \{x | g_n(x) > b\}. \qquad m(D_n) \to m(D) = \infty.$$

$$bm(D_n) \leq \int_{D_n} g_n dm \leq \int_{D_n} f_k dm \quad k \geq n$$

since $g_n \leq f_k$ for $k \geq n$. Now

$$\int f_k dm \geq \int_{D_n} f_k dm$$

since f_k is non-negative. Hence $\liminf \int f_n dm = \infty$.

b) $m(\{x:\phi(x)>0\})<\infty$. Choose $\epsilon>0$ so that it is less than the minimum of the positive values taken on by ϕ and set

$$\phi_\epsilon(x)=\begin{cases}\phi(x)-\epsilon & \text{if} \quad \phi(x)>0\\ 0 & \text{if} \quad \phi(x)=0.\end{cases}$$

Let

$$C_n:=\{x|g_n(x)\geq\phi_\epsilon\}$$

and

$$C=\{x:f(x)\geq\phi_\epsilon\}.$$

Then $C_n\nearrow C$. We have

$$\begin{aligned}\int_{C_n}\phi_\epsilon dm \quad\leq\quad \int_{C_n}g_n dm &\leq \int_{C_n}f_k dm \quad k\geq n\\ &\leq \int_C f_k dm \quad k\geq n\\ &\leq \int f_k dm \quad k\geq n.\end{aligned}$$

So

$$\int_{C_n} \phi_\epsilon dm \le \liminf \int f_k dm.$$

We will next let $n \to \infty$: Let c_i be the non-zero values of ϕ_ϵ so

$$\phi_\epsilon = \sum c_i \mathbf{1}_{B_i}$$

for some measurable sets $B_i \subset C$. Then

$$\int_{C_n} \phi_\epsilon dm = \sum c_i m(B_i \cap C_n) \to \sum c_i m(B_i) = \int \phi_\epsilon dm$$

since $(B_i \cap C_n) \nearrow B_i \cap C = B_i$. So

$$\int \phi_\epsilon dm \le \liminf \int f_k dm.$$

Now

$$\int \phi_\epsilon dm = \int \phi dm - \epsilon m\left(\{x|\phi(x) > 0\}\right).$$

Since we are assuming that $m\left(\{x|\phi(x) > 0\}\right) < \infty$, we can let $\epsilon \to 0$ and conclude that $\int \phi dm \le \liminf \int f_k dm$. QED

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The monotone convergence theorem.

We assume that $\{f_n\}$ is a sequence of non-negative measurable functions, and that $f_n(x)$ is an increasing sequence for each x. Define $f(x)$ to be the limit (possibly $+\infty$) of this sequence. We describe this situation by $f_n \nearrow f$. The monotone convergence theorem asserts that:

$$f_n \geq 0, \quad f_n \nearrow f \quad \Rightarrow \quad \lim_{n\to\infty} \int f_n dm = \int f dm. \tag{19}$$

The f_n are increasing and all $\leq f$ so the $\int f_n dm$ are monotone increasing and all $\leq \int f dm$. So the limit exists and is $\leq \int f dm$. On the other hand, Fatou's lemma gives

$$\int f dm \leq \liminf \int f_n dm = \lim \int f_n dm.$$

QED

In the monotone convergence theorem we need only know that

$$f_n \nearrow f \quad \text{a.e.}$$

Indeed, let C be the set where convergence holds, so $m(C^c) = 0$. Let $g_n = \mathbf{1}_C f_n$ and $g = \mathbf{1}_C f$. Then $g_n \nearrow g$ everywhere, so we may apply (19) to g_n and g. But $\int g_n dm = \int f_n dm$ and $\int g dm = \int f dm$ so the theorem holds for f_n and f as well.

The space $\mathcal{L}_1(X, \mathbf{R})$.

We will say an $\mathbf{R}$ valued measurable function is **integrable** if both $\int f^+ dm < \infty$ and $\int f^- dm < \infty$. If this happens, we set

$$\int f dm := \int f^+ dm - \int f^- dm. \tag{20}$$

Since both numbers on the right are finite, this difference makes sense. Some authors prefer to allow one or the other numbers (but not both) to be infinite, in which case the right hand side of (20) might be $= \infty$ or $-\infty$. We will stick with the above convention.

We will denote the set of all (real valued) integrable functions by $\mathcal{L}_1$ or $\mathcal{L}_1(X)$ or $\mathcal{L}_1(X, \mathbf{R})$ depending on how precise we want to be.

Notice that if $f \leq g$ then $f^+ \leq g^+$ and $f^- \geq g^-$ all of these functions being non-negative. So

$$\int f^+ dm \leq \int g^+ dm, \quad \int f^- dm \geq \int g^- dm$$

hence

$$\int f^+ dm - \int f^- dm \leq \int g^+ dm - \int g^- dm$$

or

$$f \leq g \quad \Rightarrow \quad \int f dm \leq \int g dm. \tag{21}$$

If a is a non-negative number, then $(af)^\pm = af^\pm$. If $a < 0$ then $(af)^\pm = (-a)f^\mp$ so in all cases we have

$$\int af dm = a \int f dm. \tag{22}$$

Additivity of the integral.

We now wish to establish

$$f, g \in \mathcal{L}_1 \ \Rightarrow f + g \in \mathcal{L}_1 \ \text{ and } \ \int (f+g)dm = \int f dm + \int g dm. \tag{23}$$

Proof. We prove this in stages:

- First assume $f = \sum a_i \mathbf{1}_{A_i}$, $g = \sum b_i \mathbf{1}_{B_i}$ are non-negative simple functions, where the A_i partition X as do the B_j. Then we can decompose and recombine the sets to yield:

$$\begin{aligned}
\int (f+g)dm &= \sum_{i,j} (a_i + b_j) m(A_i \cap B_j) \\
&= \sum_i \sum_j a_i m(A_i \cap B_j) + \sum_j \sum_i b_j m(A_i \cap B_j) \\
&= \sum_i a_i m(A_i) + \sum_j b_j m(B_j) \\
&= \int f dm + \int g dm
\end{aligned}$$

where we have used the fact that m is additive and the $A_i \cap B_j$ are disjoint sets whose union over j is A_i and whose union over i is B_j.

- Next suppose that f and g are non-negative measurable functions with finite integrals. Set

$$f_n := \sum_{k=0}^{2^{2n}} \frac{k}{2^n} \mathbf{1}_{f^{-1}[\frac{k}{2^n}, \frac{k+1}{2^n}]}.$$

Each f_n is a simple function $\leq f$, and passing from f_n to f_{n+1} involves splitting each of the sets $f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}])$ in the sum into two, and choosing a larger value on the second portion. So the f_n are increasing. Also, if $f(x) < \infty$, then $f(x) < 2^m$ for some m, and for any $n > m$ $f_n(x)$ differs from $f(x)$ by at most 2^{-n}. Hence $f_n \nearrow f$ a.e., since f is finite a.e because its integral is finite. Similarly we can construct $g_n \nearrow g$. Also $(f_n + g_n) \nearrow f + g$ a.e.

By the a.e. monotone convergence theorem

$$\int (f+g)dm = \lim \int (f_n+g_n)dm =$$

$$\lim \int f_n dm + \lim \int g_n dm = \int f dm + \int g dm,$$

By the a.e. monotone convergence theorem

$$\int (f+g)dm = \lim \int (f_n+g_n)dm = \lim \int f_n dm + \lim \int g_n dm = \int f dm + \int g dm,$$

where we have used (23) for simple functions. This argument shows that $\int (f+g)dm < \infty$ if both integrals $\int f dm$ and $\int g dm$ are finite.

For any $f \in \mathcal{L}_1$ we conclude from the preceding that

$$\int |f| dm = \int (f^+ + f^-)dm < \infty.$$

- For any $f \in \mathcal{L}_1$ we conclude from the preceding that

$$\int |f| dm = \int (f^+ + f^-) dm < \infty.$$

Similarly for g. Since $|f + g| \leq |f| + |g|$ we conclude that both $(f+g)^+$ and $(f+g)^-$ have finite integrals. Now

$$(f+g)^+ - (f+g)^- = f + g = (f^+ - f^-) + (g^+ - g^-)$$

or

$$(f+g)^+ + f^- + g^- = f^+ + g^+ + (f+g)^-.$$

All expressions are non-negative and integrable. So integrate both sides to get (23).QED

We have thus established

Theorem 2 *The space $\mathcal{L}_1(X, \mathbf{R})$ is a real vector space and $f \mapsto \int f dm$ is a linear function on $\mathcal{L}_1(X, \mathbf{R})$.*

We also have

Proposition 3 *If $h \in \mathcal{L}_1$ and $\int_A h dm \geq 0$ for all $A \in \mathcal{F}$ then $h \geq 0$ a.e.*

Proof: Let $A_n : \{x | h(x) \leq -\frac{1}{n}\}$. Then

$$\int_{A_n} h dm \leq \int_{A_n} \frac{-1}{n} dm = -\frac{1}{n} m(A_n)$$

so $m(A_n) = 0$. But if we let $A := \{x | h(x) < 0\}$ then $A_n \nearrow A$ and hence $m(A) = 0$.QED

We have defined the integral of any function f as $\int f dm = \int f^+ dm - \int f^- dm$, and $\int |f| dm = \int f^+ dm + \int f^- dm$. Since for any two non-negative real numbers $a - b \leq a + b$ we conclude that

$$\left| \int f dm \right| \leq \int |f| dm. \tag{24}$$

If we define

$$\|f\|_1 := \int |f| dm$$

we have verified that

$$\|f + g\|_1 \leq \|f\|_1 + \|g\|_1,$$

and have also verified that

$$\|cf\|_1 = |c| \|f\|_1.$$

$$\|f + g\|_1 \le \|f\|_1 + \|g\|_1,$$

and have also verified that

$$\|cf\|_1 = |c|\|f\|_1.$$

In other words, $\|\cdot\|_1$ is a semi-norm on $\mathcal{L}_1$. From the preceding proposition we know that $\|f\|_1 = 0$ if and only if $f = 0$ a.e. The question of whether we want to pass to the quotient and identify two functions which differ on a set of measure zero is a matter of taste.

The dominated convergence theorem.

This says that

Theorem 3 *Let f_n be a sequence of measurable functions such that*

$$|f_n| \leq g \ a.e., \quad g \in \mathcal{L}_1.$$

Then

$$f_n \to f \quad a.e. \ \Rightarrow \ f \in \mathcal{L}_1 \ and \int f_n dm \to \int f dm.$$

Proof. The functions f_n are all integrable, since their positive and negative parts are dominated by g. Assume for the moment that $f_n \geq 0$. Then Fatou's lemma says that

$$\int f dm \leq \liminf \int f_n dm.$$

$f_n \geq 0$. Then Fatou's lemma says that

$$\int f dm \leq \liminf \int f_n dm.$$

Fatou's lemma applied to $g - f_n$ says that

$$\int (g-f) dm \leq \liminf \int (g-f_n) dm = \liminf \left(\int g dm - \int f_n dm \right)$$

$$= \int g dm - \limsup \int f_n dm.$$

Subtracting $\int g dm$ gives

$$\limsup \int f_n dm \leq \int f dm.$$

So

$$\limsup \int f_n dm \leq \int f dm \leq \liminf \int f_n dm$$

which can only happen if all three are equal.

We have proved the result for non-negative f_n. For general f_n we can write our hypothesis as

$$-g \le f_n \le g \quad \text{a.e.}.$$

Adding g to both sides gives

$$0 \le f_n + g \le 2g \quad \text{a.e.}.$$

We now apply the result for non-negative sequences to $g + f_n$ and then subtract off $\int g dm$.

Riemann integrability.

Suppose that $X = [a, b]$ is an interval. What is the relation between the Lebesgue integral and the Riemann integral? Let us suppose that $[a, b]$ is bounded and that f is a bounded function, say $|f| \leq M$. Each partition

$$P : a = a_0 < a_1 < \cdots < a_n = b$$

into intervals $I_i = [a_{i-1}, a_i]$ with

$$m_i := m(I_i) = a_i - a_{i-1}, \quad i = 1, \ldots, n$$

defines a Riemann lower sum

$$L_P = \sum k_i m_i \quad k_i = \inf_{x \in I_i} f(x)$$

and a Riemann upper sum

$$U_P = \sum M_i m_i \quad M_i := \sup_{x \in I_i} f(x)$$

$$L_P = \sum k_i m_i \quad k_i = \inf_{x \in I_i} f(x) \qquad U_P = \sum M_i m_i \quad M_i := \sup_{x \in I_i} f(x)$$

which are the Lebesgue integrals of the simple functions

$$\ell_P := \sum k_i \mathbf{1}_{I_i} \quad \text{and } u_P := \sum M_i \mathbf{1}_{I_i}$$

respectively.

According to Riemann, we are to choose a sequence of partitions P_n which refine one another and whose maximal interval lengths go to zero. Write ℓ_i for ℓ_{P_i} and u_i for u_{P_i}. Then

$$\ell_1 \leq \ell_2 \leq \cdots \leq f \leq \cdots \leq u_2 \leq u_1.$$

Suppose that f is measurable. All the functions in the above inequality are Lebesgue integrable, so dominated convergence implies that

$$\lim U_n = \lim \int_a^b u_n dx = \int_a^b u dx$$

where $u = \lim u_n$ with a similar equation for the lower bounds. The Riemann integral is defined as the common value of $\lim L_n$ and $\lim U_n$ whenever these limits are equal.

Proposition 4 *f is Riemann integrable if and only if f is continuous almost everywhere.*

Proof. Notice that if x is not an endpoint of any interval in the partitions, then f is continuous at x if and only if $u(x) = \ell(x)$. Riemann's condition for integrability says that $\int (u - \ell) dm = 0$ which implies that f is continuous almost everywhere.

Conversely, if f is continuous a.e. then $u = f = \ell$ a.e.. Since u is measurable so is f, and since we are assuming that f is bounded, we conclude that f Lebesgue integrable. As $\ell = f = u$ a.e. their Lebesgue integrals coincide. But the statement that the Lebesgue integral of u is the same as that of ℓ is precisely the statement of Riemann integrability.QED

Notice that in the course of the proof we have also shown that the Lebesgue and Riemann integrals coincide when both exist.