Math 212 Lecture 12

The Lebesgue integral, 2.

The Beppo-Levi theorem.

Lemma 1 *Let $\{g_n\}$ be a sequence of non-negative measurable functions. Then*

$$\int \sum_{n=1}^{\infty} g_n \, dm = \sum_{n=1}^{\infty} \int g_n dm.$$

Proof. We have

$$\int \sum_{k=1}^{n} g_k dm = \sum_{k=1}^{n} \int g_k dm$$

for finite n by the linearity of the integral. Since the $g_k \geq 0$, the sums under the integral sign are increasing, and by definition converge to $\sum_{k=1}^{\infty} g_k$. The monotone convergence theorem implies the lemma. QED

But both sides of the equation in the lemma might be infinite.

Theorem 4 Beppo-Levi. *Let* $f_n \in \mathcal{L}_1$ *and suppose that*

$$\sum_{k=1}^{\infty} \int |f_k| dm < \infty.$$

Then $\sum f_k(x)$ *converges to a finite limit for almost all* x, *the sum is integrable, and*

$$\int \sum_{k=1}^{\infty} f_k \, dm = \sum_{k=1}^{\infty} \int f_k dm.$$

Proof. Take $g_n := |f_n|$ in the lemma. If we set $g = \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} |f_n|$ then the lemma says that

$$\int g dm = \sum_{n=1}^{\infty} \int |f_n| dm,$$

and we are assuming that this sum is finite. So g is integrable, in particular the set of x for which $g(x) = \infty$ must have measure zero. In other words,

$$\sum_{n=1} |f_n(x)| < \infty \quad \text{a.e. } .$$

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If a series is absolutely convergent, then it is convergent, so we can say that $\sum f_n(x)$ converges almost everywhere. Let

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

at all points where the series converges, and set $f(x) = 0$ at all other points. Now

$$\left| \sum_{n=0}^{\infty} f_n(x) \right| \leq g(x)$$

at all points, and hence by the dominated convergence theorem, $f \in \mathcal{L}_1$ and

$$\int f dm = \int \lim_{n \to \infty} \sum_{k=1}^{n} f_k \ dm = \lim_{n \to \infty} \sum \int f_k dm = \sum_{k=1}^{\infty} \int f_k dm$$

QED

$\mathcal{L}_1$ is complete.

This is an immediate corollary of the Beppo-Levi theorem and Fatou's lemma. Indeed, suppose that $\{h_n\}$ is a Cauchy sequence in $\mathcal{L}_1$. Choose n_1 so that

$$\|h_n - h_{n_1}\| \leq \frac{1}{2} \quad \forall\, n \geq n_1.$$

Then choose $n_2 > n_1$ so that

$$\|h_n - h_{n_2}\| \leq \frac{1}{2^2} \quad \forall\, n \geq n_2.$$

Continuing this way, we have produced a subsequence h_{n_j} such that

$$\|h_{n_{j+1}} - h_{n_j}\| \leq \frac{1}{2^j}.$$

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Let

$$f_j := h_{n_{j+1}} - h_{n_j}.$$

Then

$$\int |f_j| dm < \frac{1}{2^j}$$

so the hypotheses of the Beppo-Levy theorem are satisfied, and $\sum f_j$ converges almost everywhere to some limit $f \in \mathcal{L}_1$. But

$$h_{n_1} + \sum_{j=1}^{k} f_j = h_{n_{k+1}}.$$

So the subsequence h_{n_k} converges almost everywhere to some $h \in \mathcal{L}_1$.

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So the subsequence h_{n_k} converges almost everywhere to some $h \in \mathcal{L}_1$.

We must show that this h is the limit of the h_n in the $\| \cdot \|_1$ norm. For this we will use Fatou's lemma.

For a given $\epsilon > 0$, choose N so that $\|h_n - h_m\| < \epsilon$ for $k, n > N$. Since $h = \lim h_{n_j}$ we have, for $k > N$,

$$\|h - h_k\|_1 = \int |h - h_k| dm = \int \lim_{j \to \infty} |h_{n_j} - h_k| dm \leq \liminf \int |h_{n_j} - h_k| dm$$

$$= \liminf \|h_{n_j} - h_k\| < \epsilon.$$

QED

Dense subsets of $\mathcal{L}_1(\mathbf{R}, \mathbf{R})$.

Up until now we have been studying integration on an arbitrary measure space $(X, \mathcal{F}, m)$. In this section and the next, we will

take $X = \mathbf{R}$, $\mathcal{F}$ to be the σ-field of Lebesgue measurable sets, and m to be Lebesgue measure, in order to simplify some of the formulations and arguments.

Suppose that f is a Lebesgue integrable non-negative function on $\mathbf{R}$. We know that for any $\epsilon > 0$ there is a simple function ϕ such that

$$\phi \leq f$$

and

$$\int f dm - \int \phi dm = \int (f - \phi) dm = \|f - \phi\|_1 < \epsilon.$$

To say that ϕ is simple implies that

$$\phi = \sum a_i \mathbf{1}_{A_i}$$

(finite sum) where each of the $a_i > 0$ and since $\int \phi dm < \infty$ each A_i has finite measure. Since $m(A_i \cap [-n, n]) \to m(A_i)$ as $n \to \infty$, we may choose n sufficiently large so that

$$\|f - \psi\|_1 < 2\epsilon \quad \text{where} \quad \psi = \sum a_i \mathbf{1}_{A_i \cap [-n,n]}.$$

For each of the sets $A_i \cap [-n, n]$ we can find a bounded open set U_i which contains it, and such that $m(U_i/A_i)$ is as small as we please. So we can find finitely many bounded open sets U_i such that

$$\|f - \sum a_i \mathbf{1}_{U_i}\|_1 < 3\epsilon.$$

$$\|f - \sum a_i \mathbf{1}_{U_i}\|_1 < 3\epsilon.$$

Each U_i is a countable union of disjoint open intervals, $U_i = \bigcup_j I_{i,j}$, and since $m(U_i) = \sum_j m(I_{i,j})$, we can find finitely many $I_{i,j}$, j ranging over a finite set of integers, J_i such that $m\left(\bigcup_{j \in J_i}\right)$ is as close as we like to $m(U_i)$. So let us call a **step function** a function of the form $\sum b_i \mathbf{1}_{I_i}$ where the I_i are bounded intervals. We have shown that we can find a step function with positive coefficients which is as close as we like in the $\|\cdot\|_1$ norm to f. If f is not necessarily non-negative, we know (by definition!) that f^+ and f^- are in $\mathcal{L}_1$, and so we can approximate each by a step function. the triangle inequality then gives

Proposition 5 *The step functions are dense in* $\mathcal{L}_1(\mathbf{R}, \mathbf{R})$.

Proposition 5 *The step functions are dense in $\mathcal{L}_1(\mathbf{R}, \mathbf{R})$.*

If $[a, b]$, $a < b$ is a finite interval, we can approximate $\mathbf{1}_{[a,b]}$ as closely as we like in the $\|\cdot\|_1$ norm by continuous functions: just choose n large enough so that $\frac{2}{n} < b - a$, and take the function which is 0 for $x < a$, rises linearly from 0 to 1 on $[a, a + \frac{1}{n}]$, is identically 1 on $[a + \frac{1}{n}, b - \frac{1}{n}]$, and goes down linearly from 1 to 0 from $b - \frac{1}{n}$ to b and stays 0 thereafter. As $n \to \infty$ this clearly tends to $\mathbf{1}_{[a,b]}$ in the $\|\cdot\|_1$ norm. So

Proposition 6 *The continuous functions of compact support are dense in $\mathcal{L}_1(\mathbf{R}, \mathbf{R})$.*

As a consequence of this proposition, we see that we could have avoided all of measure theory if our sole purpose was to define the space $\mathcal{L}_1(\mathbf{R}, \mathbf{R})$. We could have defined it to be the completion of the space of continuous functions of compact support relative to the $\|\cdot\|_1$ norm.

The Riemann-Lebesgue Lemma.

We will state and prove this in the "generalized form". Let h be a bounded measurable function on $\mathbf{R}$. We say that h satisfies the **averaging condition** if

$$\lim_{|c|\to\infty} \frac{1}{|c|}\int_0^c h dm \to 0. \tag{25}$$

For example, if $h(t) = \cos \xi t,\ \xi \neq 0$, then the expression under the limit sign in the averaging condition is

$$\frac{1}{c\xi}\sin \xi t$$

which tends to zero as $|c| \to \infty$. Here the oscillations in h are what give rise to the averaging condition. As another example, let

$$h(t) = \begin{cases} 1 & |t| \leq t \\ 1/|t| & |t| \geq 1. \end{cases}$$

$$\lim_{|c|\to\infty} \frac{1}{|c|} \int_0^c h dm \to 0. \tag{25}$$

let

$$h(t) = \begin{cases} 1 & |t| \leq t \\ 1/|t| & |t| \geq 1. \end{cases}$$

Then the left hand side of (25) is

$$\frac{1}{|c|}(1 + \log |c|), \qquad |c| \geq 1.$$

Here the averaging condition is satisfied because the integral in (25) grows more slowly that $|c|$.

Theorem 5 [Generalized Riemann-Lebesgue Lemma].

Let $f \in \mathcal{L}_1([c,d],\mathbf{R})$, $-\infty \le c < d \le \infty$. *If* h *satisfies the averaging condition (25) then*

$$\lim_{r\to\infty}\int_c^d f(t)h(rt)dt = 0. \tag{26}$$

Proof. Our proof will use the density of step functions, Proposition 5. We first prove the theorem when $f = \mathbf{1}_{[a,b]}$ is the indicator function of a finite interval. Suppose for example that $0 \le a < b$.Then the integral on the right hand side of (26) is

$$\begin{aligned}\int_0^\infty \mathbf{1}_{[a.b]}h(rt)dt &= \int_a^b h(rt)dt, \quad \text{or setting } x = rt \\ &= \frac{1}{r}\int_0^{br} h(x)dx - \frac{1}{r}\int_0^{ra} h(x)dx\end{aligned}$$

and each of these terms tends to 0 by hypothesis. The same argument will work for any bounded interval $[a,b]$.

To prove:
$$\lim_{r\to\infty}\int_c^d f(t)h(rt)dt = 0. \qquad (26)$$

we have proved (26) for

indicator functions of intervals and hence for step functions.

Now let M be such that $|h| \le M$ everywhere (or almost everywhere) and choose a step function s so that $\|f-s\|_1 \le \frac{\epsilon}{2M}$.

Then $fh = (f-s)h + sh$

$$\begin{aligned}
\left|\int f(t)h(rt)dt\right| &= \left|\int (f(t)-s(t)h(rt)dt + \int s(t)h(rt)dt\right| \\
&\le \left|\int (f(t)-s(t))h(rt)dt\right| + \left|\int s(t)h(rt)dt\right| \\
&\le \frac{\epsilon}{2M}M + \left|\int s(t)h(rt)dt\right|.
\end{aligned}$$

We can make the second term $< \frac{\epsilon}{2}$ by choosing r large enough.
QED

The Cantor-Lebesgue theorem.

This says:

Theorem 6 *If a trigonometric series*

$$\frac{a_0}{2} + \sum_n d_n \cos(nt - \phi_n) \qquad d_n \in \mathbf{R}$$

converges on a set E of positive Lebesgue measure then

$$d_n \to 0.$$

(I have written the general form of a real trigonometric series as a cosine series with phases since we are talking about only real valued functions at the present. Of course, applied to the real and imaginary parts, the theorem asserts that if $\sum a_n e^{inx}$ converges on a set of positive measure, then the $a_n \to 0$. Also, the notation suggests - and this is my intention - that the n's are integers. But in the proof below all that we will need is that the n's are any sequence of real numbers tending to ∞.)

Proof. The proof is a nice application of the dominated convergence theorem, which was invented by Lebesgue in part precisely to prove this theorem.

We may assume (by passing to a subset if necessary) that E is contained in some finite interval $[a,b]$. If $d_n \not\to 0$ then there is an $\epsilon > 0$ and a subsequence $|d_{n_k}| > \epsilon$ for all k. If the series converges, all its terms go to 0, so this means that

$$\cos(n_k t - \phi_k) \to 0 \quad \forall\ t \in E.$$

So

$$\cos^2(n_k t - \phi_k) \to 0 \quad \forall\ t \in E.$$

Now $m(E) < \infty$ and $\cos^2(n_k t - \phi_k) \leq 1$ and the constant 1 is integrable on $[a,b]$. So we may take the limit under the integral sign using the dominated convergence theorem to conclude that

$$\lim_{k\to\infty} \int_E \cos^2(n_k t - \phi_k) dt = \int_E \lim_{k\to\infty} \cos^2(n_k t - \phi_k) dt = 0.$$

$$\lim_{k\to\infty}\int_E \cos^2(n_k t - \phi_k)dt = \int_E \lim_{k\to\infty}\cos^2(n_k t - \phi_k)dt = 0.$$

But

$$\cos^2(n_k t - \phi_k) = \frac{1}{2}[1 + \cos 2(n_k t - \phi_k)]$$

so

$$\begin{aligned}
\int_E \cos^2(n_k t - \phi_k)dt &= \frac{1}{2}\int_E [1 + \cos 2(n_k t - \phi_k)]dt \\
&= \frac{1}{2}\left[m(E) + \int_E \cos 2(n_k t - \phi_k)\right] \\
&= \frac{1}{2}m(E) + \frac{1}{2}\int_{\mathbf{R}} \mathbf{1}_E \cos 2(n_k t - \phi_k)dt.
\end{aligned}$$

But $\mathbf{1}_E \in \mathcal{L}_1(\mathbf{R}, \mathbf{R})$ so the second term on the last line goes to 0 by the Riemann Lebesgue Lemma. So the limit is $\frac{1}{2}m(E)$ instead of 0, a contradiction. QED

Fubini's theorem.

This famous theorem asserts that under suitable conditions, a double integral is equal to an iterated integral. We will prove it for real (and hence finite dimensional) valued functions on arbitrary measure spaces. (The proof for Banach space valued functions is a bit more tricky, and we shall omit it as we will not need it. This is one of the reasons why we have developed the real valued theory first.) We begin with some facts about product σ-fields.

Guido Fubini

Born: 19 Jan 1879 in Venice, Italy
Died: 6 June 1943 in New York, USA

Product σ-fields.

Let $(X,\mathcal{F})$ and $(Y,\mathcal{G})$ be spaces with σ-fields. On $X\times Y$ we can consider the collection $\mathcal{P}$ of all sets of the form

$$A\times B,\quad A\in\mathcal{F},\ B\in\mathcal{G}.$$

The σ-field generated by $\mathcal{P}$ will, by abuse of language, be denoted by

$$\mathcal{F}\times\mathcal{G}.$$

If E is any subset of $X\times Y$, by an even more serious abuse of language we will let

$$E_x:=\{y|(x,y)\in E\}$$

and (contradictorily) we will let

$$E_y:=\{x|(x,y)\in E\}.$$

The set E_x will be called the x-**section** of E and the set E_y will be called the y-section of E.

Finally we will let $\mathcal{C} \subset \mathcal{P}$ denote the collection of **cylinder sets**, that is sets of the form

$$A \times Y \qquad A \in \mathcal{F}$$

or

$$X \times B, \qquad B \in \mathcal{G}.$$

In other words, an element of $\mathcal{P}$ is a cylinder set when one of the factors is the whole space.

Theorem 7 .

- $\mathcal{F} \times \mathcal{G}$ *is generated by the collection of cylinder sets* $\mathcal{C}$.
- $\mathcal{F} \times \mathcal{G}$ *is the smallest* σ*-field on* $X \times Y$ *such that the projections*

$$\begin{aligned} \mathrm{pr}_X : X \times Y \to X \qquad & \mathrm{pr}_X(x, y) = x \\ \mathrm{pr}_Y : X \times Y \to Y \qquad & \mathrm{pr}_Y(x, y) = y \end{aligned}$$

  are measurable maps.
- *For each* $E \in \mathcal{F} \times \mathcal{G}$ *and all* $x \in X$ *the* x*-section* E_x *of* E *belongs to* $\mathcal{F}$ *and for all* $y \in Y$ *the* y*-section* E_y *of* E *belongs to* $\mathcal{G}$.

Proof. $A \times B = (A \times Y) \cap (X \times B)$ so any σ-field containing $\mathcal{C}$ must also contain $\mathcal{P}$. This proves the first item.

Since $\mathrm{pr}_X^{-1}(A) = A \times Y$, the map pr_X is measurable, and similarly for Y. But also, any σ-field containing all $A \times Y$ and $X \times B$ must contain $\mathcal{P}$ by what we just proved. This proves the second item.

As to the third item, any set E of the form $A \times B$ has the desired section properties, since its x section is B if $x \in A$ or the empty set if $x \notin A$. Similarly for its y sections. So let $\mathcal{H}$ denote the collection of subsets E which have the property that all $E_x \in \mathcal{G}$ and all $E_y \in \mathcal{F}$. If we show that $\mathcal{H}$ is a σ-field we are done.

Now $E_x^c = (E_x)^c$ and similarly for y, so $\mathcal{G}$ is closed under taking complements. Similarly for countable unions:

$$\left(\bigcup_n E_n\right)_x = \bigcup_n (E_n)_x.$$

QED

π-systems

Recall that the σ-field $\sigma(\mathcal{C})$ generated by a collection $\mathcal{C}$ of subsets of X is the intersection of all the σ-fields containing $\mathcal{C}$. Sometimes the collection $\mathcal{C}$ is closed under finite intersection. In that case, we call $\mathcal{C}$ a π-system. Examples:

- X is a topological space, and $\mathcal{C}$ is the collection of open sets in X.
- $X = \mathbf{R}$, and $\mathcal{C}$ consists of all half infinite intervals of the form $(-\infty, a]$. We will denote this π system by $\pi(\mathbf{R})$.

λ-systems

A collection $\mathcal{H}$ of subsets of X will be called a λ-system if

1. $X \in \mathcal{H}$,
2. $A, B \in \mathcal{H}$ with $A \cap B = \emptyset \Rightarrow \ A \cup B \in \mathcal{H}$,
3. $A, B \in \mathcal{H}$ and $B \subset A \Rightarrow \ (A \setminus B) \in \mathcal{H}$, and
4. $\{A_n\}_1^\infty \subset \mathcal{H}$ and $A_n \nearrow A \ \Rightarrow A \in \mathcal{H}$.

From items 1) and 3) we see that a λ-system is closed under complementation, and since $\emptyset = X^c$ it contains the empty set. If $\mathcal{B}$ is both a π-system and a λ system, it is closed under any finite union, since $A \cup B = A \cup (B/(A \cap B)$ which is a disjoint union. Any countable union union can be written in the form $A = \nearrow A_n$ where the A_n are finite disjoint unions as we have already argued. So we have proved

Proposition 7 *If $\mathcal{H}$ is both a π-system and a λ-system then it is a σ-field.*

Dynkin's lemma.

Proposition 8 [Dynkin's lemma.] *If $\mathcal{C}$ is a π-system, then the σ-field generated by $\mathcal{C}$ is the smallest λ-system containing $\mathcal{C}$.*

Let $\mathcal{M}$ be the σ-field generated by $\mathcal{C}$, and $\mathcal{H}$ the smallest λ-system containing $\mathcal{C}$. So $\mathcal{M} \supset \mathcal{H}$. By the preceding proposition, all we need to do is show that $\mathcal{H}$ is a π-system.

Let

$$\mathcal{H}_1 := \{A | \ A \cap C \in \mathcal{H} \ \forall C \in \mathcal{C}\}.$$

Clearly $\mathcal{H}_1$ is a λ-system containing $\mathcal{C}$, so $\mathcal{H} \subset \mathcal{H}_1$ which means that $A \cap C \in \mathcal{H}$ for all $A \in \mathcal{H}$ and $C \in \mathcal{C}$.

Let

$$\mathcal{H}_2 := \{A | \ A \cap H \in \mathcal{H} \quad \forall H \in \mathcal{H}\}.$$

$\mathcal{H}_2$ is again a λ-system, and it contains $\mathcal{C}$ by what we have just proved. So $\mathcal{H}_2 \supset \mathcal{H}$, which means that the intersection of two elements of $\mathcal{H}$ is again in $\mathcal{H}$, i.e. $\mathcal{H}$ is a π-system. QED

The monotone class theorem.

Theorem 8 *Let* $\mathbf{B}$ *be a class of bounded real valued functions on a space* Z *satisfying*

1. $\mathbf{B}$ *is a vector space over* $\mathbf{R}$.
2. *The constant function* $\mathbf{1}$ *belongs to* $\mathbf{B}$.
3. $\mathbf{B}$ *contains the indicator functions* $\mathbf{1}_A$ *for all* A *belonging to a* π*-system* $\mathcal{I}$.
4. *If* $\{f_n\}$ *is a sequence of non-negative functions in* $\mathbf{B}$ *and* $f_n \nearrow f$ *where* f *is a bounded function on* Z*, then* $f \in \mathbf{B}$.

Then $\mathbf{B}$ *contains every bounded* $\mathcal{M}$ *measurable function, where* $\mathcal{M}$ *is the* σ*-field generated by* $\mathcal{I}$.

Proof. Let $\mathcal{H}$ denote the class of subsets of Z whose indicator functions belong to $\mathbf{B}$. Then $Z \in \mathcal{H}$ by item 2). If $B \subset A$ are both in $\mathcal{H}$, then $\mathbf{1}_{A \setminus B} = \mathbf{1}_A - \mathbf{1}_B$ and so $A \setminus B$ belongs to $\mathcal{H}$ by item 1). Similarly, if $A \cap B = \emptyset$ then $\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B$ and so if A and B belong to $\mathcal{H}$ so does $A \cup B$ when $A \cap B = \emptyset$. Finally, condition 4) in the theorem implies condition 4) in the definition of a λ-system. So we have proved that that $\mathcal{H}$ is a λ-system containing $\mathcal{I}$. So by Dynkin's lemma, it contains $\mathcal{M}$.

Now suppose that $0 \leq f \leq K$ is a bounded $\mathcal{M}$ measurable function, where we may take K to be an integer. For each integer $n \geq 0$ divide the interval $[0, K]$ up into subintervals of size 2^{-n}, and let

$$A(n,i) := \{z | i2^{-n} \leq f(z) < (i+1)2^{-n}\}$$

where i ranges from 0 to $K2^n$. Let

$$s_n(z) := \sum_{i=0}^{K2^n} \frac{i}{2^n} \mathbf{1}_{A(n,i)}.$$

Since f is assumed to be $\mathcal{M}$-measurable, each $A(n,i) \in \mathcal{M}$, so by the preceding, and condition 1), $f_n \in \mathbf{B}$. But $0 \leq s_n \nearrow f$, and hence by condition 4), $f \in B$.

For a general bounded $\mathcal{M}$ measurable f, both f^+ and f^- are bounded and $\mathcal{M}$ measurable, and hence by the preceding and condition 1), $f = f^+ - f^- \in \mathbf{B}$.QED

We now want to apply the monotone class theorem to our situation of a product space. So $Z = X \times Y$, where $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are spaces with σ-fields, and where we take $\mathcal{I} = \mathcal{P}$ to be the π-system consisting of the product sets $A \times B, \quad A \in \mathcal{F}, \ B \in \mathcal{G}$.

Proposition 9 *Let* **B** *consist of all bounded real valued functions f on $X \times Y$ which are $\mathcal{F} \times \mathcal{G}$-measurable, and which have the property that*

- *for each $x \in X$, the function $y \mapsto f(x, y)$ is $\mathcal{G}$-measurable, and*
- *for each $y \in Y$ the function $x \mapsto f(x, y)$ is $\mathcal{F}$-measurable.*

Then **B** *consists of all bounded $\mathcal{F} \times \mathcal{G}$ measurable functions.*

Proposition 9 *Let* $\mathbf{B}$ *consist of all bounded real valued functions* f *on* $X \times Y$ *which are* $\mathcal{F} \times \mathcal{G}$*-measurable, and which have the property that*

- *for each* $x \in X$*, the function* $y \mapsto f(x, y)$ *is* $\mathcal{G}$*-measurable, and*

- *for each* $y \in Y$ *the function* $x \mapsto f(x, y)$ *is* $\mathcal{F}$*-measurable.*

Then $\mathbf{B}$ *consists of all bounded* $\mathcal{F} \times \mathcal{G}$ *measurable functions.*

Indeed, $y \mapsto \mathbf{1}_{A \times B}(x, y) = \mathbf{1}_B(y)$ if $x \in A$ and $= 0$ otherwise; and similarly for $x \mapsto \mathbf{1}_{A \times B}(x, y)$. So condition 3) of the monotone class theorem is satisfied, and the other conditions are immediate. Since $\mathcal{F} \times \mathcal{G}$ was defined to be the σ-field generated by $\mathcal{P}$, the proposition is an immediate consequence of the monotone class theorem.

Fubini for finite measures and bounded functions.

Let $(X, \mathcal{F}, m)$ and $(Y, \mathcal{G}, n)$ be measure spaces with $m(X) < \infty$ and $n(Y) < \infty$. For every bounded $\mathcal{F} \times \mathcal{G}$-measurable function f, we know that the function

$$f(x, \cdot) : y \mapsto f(x, y)$$

is bounded and $\mathcal{G}$ measurable. Hence it has an integral with respect to the measure n, which we will denote by

$$\int_Y f(x, y) n(dy).$$

This is a bounded function of x (which we will prove to be $\mathcal{F}$ measurable in just a moment). Similarly we can form

$$\int_X f(x, y) m(dx)$$

which is a function of y.

Proposition 10 *Let* $\mathbf{B}$ *denote the space of bounded* $\mathcal{F} \times \mathcal{G}$ *measurable functions such that*

- $\int_Y f(x,y)n(dy)$ *is a* $\mathcal{F}$ *measurable function on* X,
- $\int_X f(x,y)m(dx)$ *is a* $\mathcal{G}$ *measurable function on* Y *and*
- $$\int_X \left(\int_Y f(x,y)n(dy)\right) m(dx) = \int_Y \left(\int_X f(x,y)m(dx)\right) n(dy). \tag{27}$$

Then $\mathbf{B}$ *consists of all bounded* $\mathcal{F} \times \mathcal{G}$ *measurable functions.*

Proof. We have verified that the first two items hold for $\mathbf{1}_{A\times B}$. Both sides of (27) equal $m(A)n(B)$ as is clear from the proof of Proposition 9. So conditions 1-3 of the monotone class theorem are clearly satisfied, and condition 4) is a consequence of two double applications of the monotone convergence theorem. QED

Now for any $C \in \mathcal{F} \times \mathcal{G}$ we define

$$(m \times n)(C) := \int_X \left(\int_Y \mathbf{1}_C(x, y) n(dy) \right) m(dx)$$
$$= \int_Y \left(\int_X \mathbf{1}_C(x, y) m(dx) \right) n(dy), \tag{28}$$

both sides being equal on account of the preceding proposition. This measure assigns the value $m(A)n(B)$ to any set $A \times B \in \mathcal{P}$, and since $\mathcal{P}$ generates $\mathcal{F} \times \mathcal{G}$ as a sigma field, any two measures which agree on $\mathcal{P}$ must agree on $\mathcal{F} \times \mathcal{G}$. Hence $m \times n$ is the unique measure which assigns the value $m(A)n(B)$ to sets of $\mathcal{P}$.

Furthermore, we know that

$$\int_{X\times Y} f(x,y)(m\times n) = \int_X \left(\int_Y f(x,y)n(dy)\right) m(dx) = \int_Y \left(\int_X f(x,y)m(dx)\right) n(dy) \tag{29}$$

is true for functions of the form $\mathbf{1}_{A\times B}$ and hence by the monotone class theorem it is true for all bounded functions which are measurable relative to $\mathcal{F}\times\mathcal{G}$.

The above assertions are the content of Fubini's theorem for bounded measures and functions. We summarize:

Theorem 9 *Let $(X, \mathcal{F}, m)$ and $(Y, \mathcal{G}, n)$ be measure spaces with $m(X) < \infty$ and $n(Y) < \infty$. There exists a unique measure on $\mathcal{F} \times \mathcal{G}$ with the property that*

$$(m \times n)(A \times B) = m(A)n(B) \quad \forall\ A \times B \in \mathcal{P}.$$

For any bounded $\mathcal{F} \times \mathcal{G}$ measurable function, the double integral is equal to the iterated integral in the sense that (29) holds.

$$\int_{X \times Y} f(x,y)(m \times n) = \int_X \left(\int_Y f(x,y) n(dy) \right) m(dx) =$$

$$\int_Y \left(\int_X f(x,y) m(dx) \right) n(dy) \tag{29}$$

Extensions to unbounded functions and to σ-finite measures.

Suppose that we temporarily keep the condition that $m(X) < \infty$ and $n(Y) < \infty$. Let f be any non-negative $\mathcal{F} \times \mathcal{G}$-measurable function. We know that (29) holds for all bounded measurable functions, in particular for all simple functions. We know that we can find a sequence of simple functions s_n such that $s_n \nearrow f$. Hence by several applications of the monotone convergence theorem, we know that (29) is true for all non-negative $\mathcal{F} \times \mathcal{G}$-measurable functions in the sense that all three terms are infinite together, or finite together and equal. Now we have agreed to call a $\mathcal{F} \times \mathcal{G}$-measurable function f integrable if and only if f^+ and f^- have finite integrals. In this case (29) holds.

A measure space $(X, \mathcal{F}, m)$ is called σ-**finite** if $X = \bigcup_n X_n$ where $m(X_n) < \infty$. In other words, X is σ-finite if it is a countable union of finite measure spaces. As usual, we can then write X as a countable union of disjoint finite measure spaces. So if X and Y are σ-finite, we can write the various integrals that occur in (29) as sums of integrals which occur over finite measure spaces. A bit of standard argumentation shows that Fubini continues to hold in this case.

If X or Y is not σ-finite, or, even in the finite case, if f is not non-negative or $m \times n$ integrable, then Fubini need not hold. I hope to present the standard counter-examples in the problem set.