Math 212 Lecture 13

The Daniell-Stone theory of the integral.

The idea.

Daniell's idea was to take the axiomatic properties of the integral as the starting point and develop integration for broader and broader classes of functions. Then derive measure theory as a consequence. Much of the presentation here is taken from the book *Abstract Harmonic Analysis* by Lynn Loomis. Some of the lemmas, propositions and theorems indicate the corresponding sections in Loomis's book.

The Daniell Integral

Let L be a vector space of *bounded* real valued functions on a set S closed under $\wedge$ and $\vee$. For example, S might be a complete metric space, and L might be the space of continuous functions of compact support on S.

A map

$$I : L \to \mathbf{R}$$

is called an **Integral** if

1. I is linear: $I(af + bg) = aI(f) + bI(g)$
2. I is non-negative: $f \geq 0 \Rightarrow I(f) \geq 0$ or equivalently $f \geq g \Rightarrow I(f) \geq I(g)$.
3. $f_n \searrow 0 \Rightarrow I(f_n) \searrow 0$.

For example, we might take $S = \mathbf{R}^n$, $L =$ the space of continuous functions of compact support on $\mathbf{R}^n$, and I to be the Riemann integral. The first two items on the above list are clearly satisfied.

1. I is linear: $I(af + bg) = aI(f) + bI(g)$
2. I is non-negative: $f \geq 0 \Rightarrow I(f) \geq 0$ or equivalently $f \geq g \Rightarrow I(f) \geq I(g)$.
3. $f_n \searrow 0 \Rightarrow I(f_n) \searrow 0$.

For example, we might take $S = \mathbf{R}^n$, $L =$ the space of continuous functions of compact support on $\mathbf{R}^n$, and I to be the Riemann integral. The first two items on the above list are clearly satisfied. As to the third, we recall Dini's lemma from the notes on metric spaces, which says that a sequence of continuous functions of compact support $\{f_n\}$ on a metric space which satisfies $f_n \searrow 0$ actually converges uniformly to 0. Furthermore the supports of the f_n are all contained in a fixed compact set - for example the support of f_1. This establishes the third item.

The plan is now to successively increase the class of functions on which the integral is defined.

Define

$$U := \{\text{limits of monotone non-decreasing sequences of elements of } L\}.$$

We will use the word "increasing" as synonymous with "monotone non-decreasing" so as to simplify the language.

Lemma 1.1 *If f_n is an increasing sequence of elements of L and if $k \in L$ satisfies $k \leq \lim f_n$ then $\lim I(f_n) \geq I(k)$.*

Proof. If $k \in L$ and $\lim f_n \geq k$, then

$$f_n \wedge k \leq k \text{ and } f_n \geq f_n \wedge k$$

so $I(f_n) \geq I(f_n \wedge k)$ while

$$[k - (f_n \wedge k)] \searrow 0$$

so

$$I([k - f_n \wedge k]) \searrow 0$$

by 3) or

$$I(f_n \wedge k) \nearrow I(k).$$

Hence $\lim I(f_n) \geq \lim I(f_n \wedge k) = I(k)$. QED

Lemma 1.2 **[12C]** *If* $\{f_n\}$ *and* $\{g_n\}$ *are increasing sequences of elements of* L *and* $\lim g_n \le \lim f_n$ *then* $\lim I(g_n) \le \lim I(f_n)$.

Proof. Fix m and take $k = g_m$ in the previous lemma. Then $I(g_m) \le \lim I(f_n)$. Now let $m \to \infty$. QED

Thus

$$f_n \nearrow f \text{ and } g_n \nearrow f \quad \Rightarrow \lim I(f_n) = \lim I(g_n)$$

so we may extend I to U by setting

$$I(f) := \lim I(f_n) \quad \text{for} \quad f_n \nearrow f.$$

If $f \in L$, this coincides with our original I, since we can take $g_n = f$ for all n in the preceding lemma.

We have now extended I from L to U. The next lemma shows that if we now start with I on U and apply the same procedure again, we do not get any further.

Lemma 1.3 [12D] *If* $f_n \in U$ *and* $f_n \nearrow f$ *then* $f \in U$ *and* $I(f_n) \nearrow I(f)$.

Proof. For each fixed n choose $g_n^m \nearrow_m f_n$. Set

$$h_n := g_1^n \vee \cdots \vee g_n^n$$

so

$$h_n \in L \quad \text{and } h_n \text{ is increasing}$$

with

$$g_i^n \leq h_n \leq f_n \quad \text{for } i \leq n.$$

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Let $n \to \infty$. Then

$$f_i \leq \lim h_n \leq f.$$

Now let $i \to \infty$. We get

$$f \leq \lim h_n \leq f.$$

So we have written f as a limit of an increasing sequence of elements of L, So $f \in U$. Also

$$I(g_i^n) \leq I(h_n) \leq I(f)$$

so letting $n \to \infty$ we get

$$I(f_i) \leq I(f) \leq \lim I(f_n)$$

so passing to the limits gives $I(f) = \lim I(f_n)$. QED

Extending to *-U* .

We have

$$I(f+g) = I(f) + I(g) \quad \text{for} \quad f, g \in U.$$

Define

$$-U := \{-f \mid f \in U\}$$

and

$$I(f) := -I(-f) \qquad f \in -U.$$

If $f \in U$ and $-f \in U$ then $I(f) + I(-f) = I(f - f) = I(0) = 0$ so $I(-f) = -I(f)$ in this case. So the definition is consistent.

$-U$ is closed under monotone decreasing limits. etc.

If $g \in -U$ and $h \in U$ with $g \leq h$ then $-g \in U$ so $h - g \in U$ and $h - g \geq 0$ so $I(h) - I(g) = I(h + (-g)) = I(h - g) \geq 0$.

I - summable functions.

A function f is called I-**summable** if for every $\epsilon > 0$, $\exists\ g \in -U,\ h \in U$ with

$$g \le f \le h, \quad |I(g)| < \infty,\ |I(h)| < \infty \ \text{ and } I(h-g) \le \epsilon.$$

For such f define

$$I(f) = \text{glb}\ I(h) = \text{lub}\ I(g).$$

If $f \in U$ take $h = f$ and $f_n \in L$ with $f_n \nearrow f$. Then $-f_n \in L \subset U$ so $f_n \in -U$. If $I(f) < \infty$ then we can choose n sufficiently large so that $I(f) - I(f_n) < \epsilon$. The space of summable functions is denoted by $\overline{L}_1$. It is clearly a vector space, and I satisfies conditions 1) and 2) above, i.e. is linear and non-negative.

Theorem 1.1 [12G] **Monotone convergence theorem.** *$f_n \in \overline{L}_1$, $f_n \nearrow f$ and $\lim I(f_n) < \infty \ \Rightarrow f \in \overline{L}_1$ and $I(f) = \lim I(f_n)$.*

Proof. Replacing f_n by $f_n - f_0$ we may assume that $f_0 = 0$. Choose

$h_n \in U$, such that $f_n - f_{n-1} \le h_n$ and $I(h_n) \le I(f_n - f_{n-1}) + \frac{\epsilon}{2^n}$.

Then

$$f_n \le \sum_1^n h_i \quad \text{and} \sum_{i=1}^n I(h_i) \le I(f_n) + \epsilon.$$

Since U is closed under monotone increasing limits,

$$h := \sum_{i=1}^\infty h_i \in U, \quad f \le h \quad \text{and } I(h) \le \lim I(f_n) + \epsilon.$$

Since $f_m \in \overline{L}_1$ we can find a $g_m \in -U$ with $I(f_m) - I(g_m) < \epsilon$ and hence for m large enough $I(h) - I(g_m) < 2\epsilon$. So $f \in \overline{L}_1$ and $I(f) = \lim I(f_n)$. QED

Monotone class theorems.

A collection of functions which is closed under monotone increasing and monotone decreasing limits is called a **monotone class**. $\mathcal{B}$ is defined to be the smallest monotone class containing L.

Lemma 4 *Let $h \leq k$. If $\mathcal{M}$ is a monotone class which contains $(g \vee h) \wedge k$ for every $g \in L$, then $\mathcal{M}$ contains all $(f \vee h) \wedge k$ for all $f \in \mathcal{B}$.*

Proof. The set of f such that $(f \vee h) \wedge k \in \mathcal{M}$ is a monotone class containing L by the distributive laws.QED

Taking $h = k = 0$ this says that the smallest monotone class containing L^+, the set of non-negative functions in L, is the set $\mathcal{B}^+$, the set of non-negative functions in $\mathcal{B}$.

Here is a series of monotone class theorem style arguments:

Theorem 2.1 *$f, g \in \mathcal{B} \;\Rightarrow af + bg \in \mathcal{B}, f \vee g \in \mathcal{B}$ and $f \wedge g \in \mathcal{B}$.*

For $f \in \mathcal{B}$, let

$$\mathcal{M}(f) := \{g \in \mathcal{B} | f + g, f \vee g, f \wedge g \in \mathcal{B}\}.$$

$\mathcal{M}(f)$ is a monotone class. If $f \in L$ it includes all of L, hence all of $\mathcal{B}$. But

$$g \in \mathcal{M}(f) \;\Leftrightarrow\; f \in \mathcal{M}(g).$$

So $L \subset \mathcal{M}(g)$ for any $g \in \mathcal{B}$, and since it is a monotone class $\mathcal{B} \subset \mathcal{M}(g)$. This says that $f, g \in \mathcal{B} \Rightarrow f + g \in \mathcal{B},\; f \wedge g \in \mathcal{B}$ and $f \vee g \in \mathcal{B}$. Similarly, let $\mathcal{M}$ be the class of functions for which $cf \in \mathcal{B}$ for all real c. This is a monotone class containing L hence contains $\mathcal{B}$. QED

Lemma 2.2 *If $f \in \mathcal{B}$ there exists a $g \in U$ such that $f \leq g$.*

Proof. The limit of a monotone increasing sequence of functions in U belongs to U. Hence the set of f for which the lemma is true is a monotone class which contains L. hence it contains $\mathcal{B}$. QED

A function f is L**-bounded** if there exists a $g \in L^+$ with $|f| \leq g$. A class $\mathcal{F}$ of functions is said to be L-monotone if $\mathcal{F}$ is closed under monotone limits of L-bounded functions.

Theorem 2.2 *The smallest L-monotone class including L^+ is $\mathcal{B}^+$.*

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Proof. Call this smallest family $\mathcal{F}$. If $g \in L^+$, the set of all $f \in \mathcal{B}^+$ such that $f \wedge g \in \mathcal{F}$ form a monotone class containing L^+, hence containing $\mathcal{B}^+$ hence equal to $\mathcal{B}^+$. If $f \in \mathcal{B}^+$ and $f \leq g$ then $f \wedge g = f \in \mathcal{F}$. So $\mathcal{F}$ contains all L bounded functions belonging to $\mathcal{B}^+$. Let $f \in \mathcal{B}^+$. by the lemma, choose $g \in U$ such that $f \leq g$, and choose $g_n \in L^+$ with $g_n \nearrow g$. Then $f \wedge g_n \leq g_n$ and so is L bounded, so $f \wedge g_n \in \mathcal{F}$. Since $(f \wedge g_n) \to f$ we see that $f \in \mathcal{F}$. So

$$\mathcal{B}^+ \subset \mathcal{F}.$$

We know that $\mathcal{B}^+$ is a monotone class, in particular an L-monotone class. Hence $\mathcal{F} = \mathcal{B}^+$. QED

The space L^1.

Define

$$L^1 := \overline{L}_1 \cap \mathcal{B}.$$

Since $\overline{L}_1$ and $\mathcal{B}$ are both closed under the lattice operations,

$$f \in L^1 \Rightarrow f^{\pm} \in L^1 \Rightarrow |f| \in L^1.$$

Theorem 2.3 *If $f \in \mathcal{B}$ then $f \in L^1 \Leftrightarrow \exists g \in L^1$ with $|f| \leq g$.*

We have proved $\Rightarrow$: simply take $g = |f|$. For the converse we may assume that $f \geq 0$ by applying the result to f^+ and f^-. The family of all $h \in \mathcal{B}^+$ such that $h \wedge g \in L^1$ is monotone and includes L^+ so includes $\mathcal{B}^+$. So $f = f \wedge g \in L^1$. QED

Extend I to all of $\mathcal{B}^+$ be setting it $= \infty$ on functions which do not belong to L^1.

Stone's axiom.

Loomis calls a set A **integrable** if $\mathbf{1}_A \in \mathcal{B}$. The monotone class properties of $\mathcal{B}$ imply that the integrable sets form a σ-field. Then define

$$\mu(A) := \int \mathbf{1}_A$$

and the monotone convergence theorem guarantees that μ is a measure.

Add **Stone's axiom**

$$f \in L \Rightarrow f \wedge \mathbf{1} \in L.$$

Then the monotone class property implies that this is true with L replaced by $\mathcal{B}$.

Theorem 3.1 *$f \in \mathcal{B}$ and $a > 0 \Rightarrow$ then*

$$A_a := \{p | f(p) > a\}$$

is an integrable set. If $f \in L^1$ then

$$\mu(A_a) < \infty.$$

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Proof. Let

$$f_n := [n(f - f \wedge a)] \wedge 1 \in \mathcal{B}.$$

Then

$$f_n(x) = \begin{cases} 1 & \text{if} & f(x) \geq a + \frac{1}{n} \\ 0 & \text{if} & f(x) \leq a \\ n(f(x) - a) & \text{if} & a < f(x) < a + \frac{1}{n} \end{cases}.$$

We have

$$f_n \nearrow \mathbf{1}_{A_a}$$

so $\mathbf{1}_{A_a} \in \mathcal{B}$ and $0 \leq \mathbf{1}_{A_a} \leq \frac{1}{a} f^+$. QED

$$A_a := \{p | f(p) > a\}$$

Theorem 3.2 *If $f \geq 0$ and A_a is integrable for all $a > 0$ then $f \in \mathcal{B}$.*

Proof. For $\delta > 1$ define

$$A_m^\delta := \{x | \delta^m < f(x) \leq \delta^{m+1}\}$$

and

$$f_\delta := \sum_m \delta^m \mathbf{1}_{A_m^\delta}.$$

Each $f_\delta \in \mathcal{B}$. Take

$$\delta_n = 2^{2^{-n}}.$$

Then each successive subdivision divides the previous one into "octaves" and $f_{\delta_m} \nearrow f$. QED

Also

$$f_\delta \le f \le \delta f_\delta$$

and

$$I(f_{\delta)} = \sum \delta^n \mu(A_m^\delta) = \int f_\delta d\mu.$$

So we have

$$I(f_\delta) \le I(f) \le \delta I(f_\delta)$$

and

$$\int f_\delta d\mu \le \int f d\mu \le \delta \int f_\delta d\mu.$$

So if either of $I(f)$ or $\int f d\mu$ is finite they both are and

$$\left| I(f) - \int f d\mu \right| \le (\delta - 1) I(f_\delta) \le (\delta - 1) I(f).$$

So

$$\int f d\mu = I(f).$$

If $f \in \mathcal{B}^+$ and $a > 0$ then

$$\{x|f(x)^a > b\} = \{x|f(x) > b^{\frac{1}{a}}\}.$$

So $f \in \mathcal{B}^+ \Rightarrow f^a \in \mathcal{B}^+$ and hence the product of two elements of $\mathcal{B}^+$ belongs to $\mathcal{B}^+$ because

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right].$$

Marshall Harvey Stone

Born: 8 April 1903 in New York, USA
Died: 9 Jan 1989 in Madras, India

Hölder, Minkowski , L^p and L^q.

The numbers $p, q > 1$ are called **conjugate** if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

This is the same as

$$pq = p + q$$

or

$$(p-1)(q-1) = 1.$$

This last equation says that if

$$y = x^{p-1}$$

then

$$x = y^{q-1}.$$

The area under the curve $y = x^{p-1}$ from 0 to a is

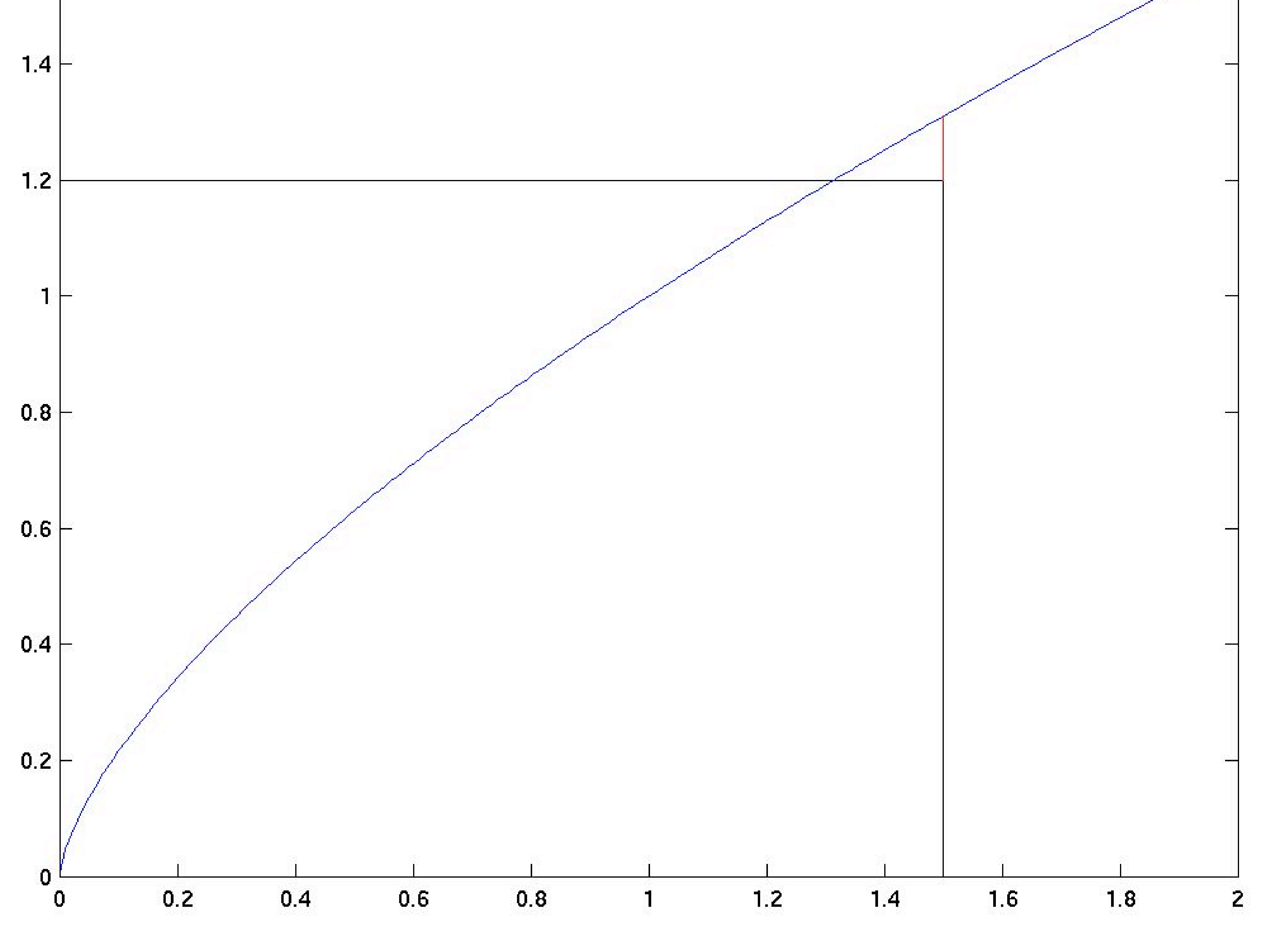
$$A = \frac{a^p}{p}$$

while the area between the same curve and the y-axis up to $y = b$

$$B = \frac{b^q}{q}.$$

Suppose $b < a^{p-1}$ to fix the ideas. Then area ab of the rectangle is less than $A + B$ or

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$



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$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

with equality if and only if $b = a^{p-1}$. Replacing a by $a^{\frac{1}{p}}$ and b by $b^{\frac{1}{q}}$ gives

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

Let L^p denote the space of functions such that $|f|^p \in L^1$. For $f \in L^p$ define

$$\|f\|_p := \left(\int |f|^p d\mu \right)^{\frac{1}{p}}.$$

We will soon see that if $p \geq 1$ this is a (semi-)norm.

$$a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

If $f \in L^p$ and $g \in L^q$ with $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$ take

$$a = \frac{|f|^p}{\|f\|_p}, \quad b = \frac{|g|^q}{\|g\|_q}$$

as functions. Then

$$\int (|f||g|)d\mu \leq \|f\|_p\|g\|_q \left(\frac{1}{p}\frac{1}{\|f\|_p^p}\int |f|^p d\mu + \frac{1}{q}\frac{1}{\|g\|_q^q}\int |g|^q d\mu\right) = \|f\|_p\|g\|$$

This shows that the left hand side is integrable and that

$$\left|\int fgd\mu\right| \leq \|f\|_p\|g\|_q \qquad (1)$$

which is known as**Hölder's inequality**. (If either $\|f||_p$ or $\|g\|_q =$ 0 then $fg = 0$ a.e. and Hölder's inequality is trivial.)
We write

$$(f, g) := \int fgd\mu.$$

Otto Ludwig Hölder

Born: 22 Dec 1859 in Stuttgart, Germany
Died: 29 Aug 1937 in Leipzig, Germany

Minkowski's inequality.

Proposition 4.1 [Minkowski's inequality] *If* $f, g \in L^p,\ p \geq 1$ *then* $f+g \in L^p$ *and*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

For $p = 1$ this is obvious. If $p > 1$

$$|f+g|^p \leq [2\max(|f|,|g|)]^p \leq 2^p\,[|f|^p + |g|^p]$$

implies that $f+g \in L^p$. Write

$$\|f+g\|_p^p \leq I(|f+g|^{p-1}|f|) + I(|f+g|^{p-1}|g|).$$

$$\|f+g\|_p^p \leq I(|f+g|^{p-1}|f|) + I(|f+g|^{p-1}|g|).$$

Now

$$q(p-1) = qp - q = p$$

so

$$|f+g|^{p-1} \in L_q$$

and its $\|\cdot\|_q$ norm is

$$I(|f+g|^p)^{\frac{1}{q}} = I(|f+q|^p)^{1-\frac{1}{p}} = I(|f+g|^p)^{\frac{p-1}{p}} = \|f+g\|_p^{p-1}.$$

So we can write the preceding inequality as

$$\|f+g\|_p^p \leq (|f|, |f+g|^{p-1}) + (|g|, |f+g|^{p-1})$$

and apply Hölder's inequality to conclude that

$$\|f+g\|^p \leq \|f+g\|^{p-1}(\|f\|_p + \|g\|_p).$$

We may divide by $\|f+g\|_p^{p-1}$ to get Minkowski's inequality unless $\|f+g\|_p = 0$ in which case it is obvious. QED

Hermann Minkowski

Born: **22 June 1864 in Alexotas, Russian Empire (now Kaunas, Lithuania)**
Died: **12 Jan 1909 in Göttingen, Germany**

L^p is complete.

Proof. Suppose $f_n \geq 0$, $f_n \in L^p$, and $\sum \|f_n\|_p < \infty$ Then

$$k_n := \sum_1^n f_j \in L^p$$

by Minkowski and since $k_n \nearrow f$ we have $|k_n|^p \nearrow f^p$ and hence by the monotone convergence theorem $f := \sum_{j=1}^{\infty} f_n \in L^p$ and $\|f\|_p = \lim \|k_n\|_p \leq \sum \|f_j\|_p$.

Now let $\{f_n\}$ be any Cauchy sequence in L^p. By passing to a subsequence we may assume that

$$\|f_{n+1} - f_n\|_p < \frac{1}{2^n}.$$

So $\sum_n^{\infty} |f_{i+1} - f_i| \in L^p$ and hence

$$g_n := f_n - \sum^{\infty} |f_{i+1} - f_i| \in L^p \text{ and } h_n := f_n + \sum^{\infty} |f_{i+1} - f_i| \in L^p.$$

$$g_n := f_n - \sum_n^\infty |f_{i+1} - f_i| \in L^p \text{ and } h_n := f_n + \sum_n^\infty |f_{i+1} - f_i| \in L^p.$$

We have

$$g_{n+1} - g_n = f_{n+1} - f_n + |f_{n+1} - f_n| \geq 0$$

so g_n is increasing and similarly h_n is decreasing. Hence $f := \lim g_n \in L^p$ and $\|f - f_n\|_p \leq \|h_n - g_n\|_p \leq 2^{-n+2} \to 0$. So the subsequence has a limit which then must be the limit of the original sequence. QED

Proposition 4.2 *L is dense in L^p for any $1 \leq p < \infty$.*

Proof. For $p = 1$ this was a defining property of L^1. More generally, suppose that $f \in L^p$ and that $f \geq 0$. Let

$$A_n := \{x : \frac{1}{n} < f(x) < n\},$$

and let

$$g_n := f \cdot \mathbf{1}_{A_n}.$$

Then $(f - g_n) \searrow 0$ as $n \to \infty$. Choose n sufficiently large so that $\|f - g_n\|_p < \epsilon/2$. Since

$$0 \leq g_n \leq n\mathbf{1}_{A_n} \quad \text{and} \quad \mu(A_n) < n^p I(|f|^p) < \infty$$

we conclude that

$$g_n \in L^1.$$

Now choose $h \in L^+$ so that

$$\|h - g_n\|_1 < \left(\frac{\epsilon}{2n}\right)^p$$

and also so that $h \leq n$. Then

$$\begin{aligned}
\|h - g_n\|_p &= (I(|h - g_n|^p))^{1/p} \\
&= \left(I(|h - g_n|^{p-1}|h - g_n|)\right)^{1/p} \\
&\leq \left(I(n^{p-1}|h - g_n|)\right)^{1/p} \\
&= \left(n^{p-1}\|h - g_n\|_1\right)^{1/p} \\
&< \epsilon/2.
\end{aligned}$$

So by the triangle inequality $\|f - h\| < \epsilon$. QED

In the above, we have not bothered to pass to the quotient by the elements of norm zero. In other words, we have not identified two functions which differ on a set of measure zero. We will continue with this ambiguity. But equally well, we could change our notation, and use L^p to denote the quotient space (as we did earlier in class) and denote the space before we pass to the quotient by $\mathcal{L}^p$ to conform with our earlier notation. I will continue to be sloppy on this point, in conformity to Loomis' notation.