Math 212 Lecture 14

The Radon-Nikodym theorem and the Riesz representation theorem.

Review: Hölder, Minkowski , L^p and L^q.

$$\|f\|_p := \left(\int |f|^p d\mu \right)^{\frac{1}{p}}$$

The numbers $p, q > 1$ are called **conjugate** if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $f \in L^p$ and $g \in L^q$

Hölder's inequality.

$$\left| \int fg d\mu \right| \le \|f\|_p \|g\|_q \qquad (1)$$

[Minkowski's inequality] *If* $f, g \in L^p,\ p \ge 1$ *then* $f + g \in L^p$ *and*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

L^p *is complete.*

$\|\cdot\|_\infty$ is the essential sup norm.

Suppose that $f \in \mathcal{B}$ has the property that it is equal almost everywhere to a function which is bounded above. We call such a function **essentially bounded** (from above). We can then define the **essential least upper bound** of f to be the smallest number which is an upper bound for a function which differs from f on a set of measure zero. If $|f|$ is essentially bounded, we denote its essential least upper bound by $\|f\|_\infty$. Otherwise we say that $\|f\|_\infty = \infty$. We let $\mathcal{L}^\infty$ denote the space of $f \in \mathcal{B}$ which have $\|f\|_\infty < \infty$. It is clear that $\|\cdot\|_\infty$ is a semi-norm on this space. The justification for this notation is

Theorem 5.1 [14G] *If $f \in L^p$ for some $p > 0$ then*

$$\|f\|_\infty = \lim_{q\to\infty} \|f\|_q. \tag{2}$$

Remark. In the statement of the theorem, both sides of (2) are allowed to be ∞.

Proof. If $\|f\|_\infty = 0$, then $\|f\|_q = 0$ for all $q > 0$ so the result is trivial in this case. So let us assume that $\|f\|_\infty > 0$ and let a be any positive number smaller that $\|f\|_\infty$. In other words,

$$0 < a < \|f\|_\infty.$$

Let

$$A_a := \|x : |f(x)| > a\}.$$

This set has positive measure by the choice of a, and its measure is finite since $f \in L^p$. Also

$$\|f\|_q \geq \left(\int_{A_a} |f|^q\right)^{1/q} \geq a\mu(A_a)^{1/q}.$$

Letting $q \to \infty$ gives

$$\liminf_{q\to\infty} \|f\|_q \geq a$$

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and since a can be any number $< \|f\|_\infty$ we conclude that

$$\liminf_{q\to\infty} \|f\|_q \geq \|f\|_\infty.$$

So we need to prove that

$$\lim \|f\|_q \leq \|f\|_\infty.$$

This is obvious if $\|f\|_\infty = \infty$. So suppose that $\|f\|_\infty$ is finite. Then for $q > p$ we have

$$|f|^q \leq |f|^p (\|f\|_\infty)^{q-p}$$

almost everywhere. Integrating and taking the q-th root gives

$$\|f\|_q \leq (\|f\|_p)^{\frac{p}{q}} (\|f\|_\infty)^{1-\frac{p}{q}}.$$

Letting $q \to \infty$ gives the desired result. QED

The Radon-Nikodym Theorem.

Suppose we are given two integrals, I and J on the same space L. That is, both I and J satisfy the three conditions of linearity, positivity, and the monotone limit property that went into our definition of the term "integral". We say that J is **absolutely continuous** with respect to I if every set which is I null (i.e. has measure zero with respect to the measure associated to I) is J null.

The integral I is said to be **bounded** if

$$I(\mathbf{1}) < \infty,$$

or, what amounts to the same thing, that

$$\mu_I(S) < \infty$$

where μ_I is the measure associated to I.

We will first formulate the Radon-Nikodym theorem for the case of bounded integrals, where there is a very clever proof due to von-Neumman which reduces it to the Riesz representation theorem in Hilbert space theory.

Theorem 6.1 [Radon-Nikodym] *Let I and J be bounded integrals, and suppose that J is absolutely continuous with respect to I. Then there exists an element $f_0 \in \mathcal{L}^1(I)$ such that*

$$J(f) = I(f f_0) \quad \forall\ f \in \mathcal{L}^1(J). \tag{3}$$

The element f_0 is unique up to equality almost everywhere (with respect to μ_I).

Proof.(After von-Neumann.) Consider the linear function

$$K := I + J$$

on L. Then K satisfies all three conditions in our definition of an integral, and in addition is bounded. We know from the case $p = 2$ of Theorem 4.1 that $L^2(K)$ is a (real) Hilbert space. (Assume for this argument that we have passed to the quotient space so an element of $L^2(K)$ is an equivalence class of of functions.) The fact that K is bounded, says that $\mathbf{1} := \mathbf{1}_S \in L^2(K)$. If $f \in L^2(K)$ then the Cauchy-Schwartz inequality says that

$$K(|f|) = K(|f| \cdot \mathbf{1}) = (|f|, \mathbf{1})_{2,K} \leq \|f\|_{2,K} \|\mathbf{1}\|_{2,K} < \infty$$

so $|f|$ and hence f are elements of $L^1(K)$.

Furthermore,

$$|J(f)| \leq J(|f|) \leq K(|f|) \leq \|f\|_{2,K} \|\mathbf{1}\|_{2,K}$$

for all $f \in L$.

$$|J(f)| \leq J(|f|) \leq K(|f|) \leq \|f\|_{2,K}\|\mathbf{1}\|_{2,K}$$

for all $f \in L$. Since we know that L is dense in $L^2(K)$ by Proposition 4.2, J extends to a unique continuous linear functional on $L^2(K)$. We conclude from the real version of the Riesz representation theorem, that there exists a unique $g \in L^2(K)$ such that

$$J(f) = (f, g)_{2,K} = K(fg).$$

If A is any subset of S of positive measure, then $J(\mathbf{1}_A) = K(\mathbf{1}_A g)$ so g is non-negative. (More precisely, g is equivalent almost everywhere to a function which is non-negative.) We obtain inductively

$$\begin{aligned} J(f) &= K(fg) = \\ I(fg) + J(fg) &= I(fg) + I(fg^2) + J(fg^2) = \\ &\vdots \\ &= I\left(f \cdot \sum_{i=1}^{n} g^i\right) + J(fg^n). \end{aligned}$$

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&\vdots \\
&= I\left(f \cdot \sum_{i=1}^{n} g^i\right) + J(fg^n).
\end{aligned}
$$

Let N be the set of all x where $g(x) \geq 1$. Taking $f = \mathbf{1}_N$ in the preceding string of equalities shows that

$$
J(\mathbf{1}_N) \geq nI(\mathbf{1}_N).
$$

Since n is arbitrary, we have proved

Lemma 6.1 *The set where $g \geq 1$ has I measure zero.*

$$
\begin{aligned}
J(f) &= K(fg) = \\
I(fg) + J(fg) &= I(fg) + I(fg^2) + J(fg^2) = \\
&\vdots \qquad\qquad \textit{The set where } g \geq 1 \textit{ has } I \textit{ measure zero.} \\
&= I\left(f \cdot \sum_{i=1}^{n} g^i\right) + J(fg^n).
\end{aligned}
$$

We have not yet used the assumption that J is absolutely continuous with respect to I. Let us now use this assumption to conclude that N is also J-null. This means that if $f \geq 0$ and $f \in L^1(J)$ then $fg^n \searrow 0$ almost everywhere (J), and hence by the dominated convergence theorem $J(fg^n) \searrow 0.$

Plugging this back into the above string of equalities shows (by the monotone convergence theorem for I) that

$$
f \sum_{i=1}^{\infty} g^n
$$

converges in the $L^1(I)$ norm to $J(f)$.

$$f \sum_{i=1}^{\infty} g^n$$

converges in the $L^1(I)$ norm to $J(f)$. In particular, since $J(\mathbf{1}) < \infty$, we may take $f = \mathbf{1}$ and conclude that $\sum_{i=1}^{\infty} g^i$ converges in $L^1(I)$. So set

$$f_0 := \sum_{i=1}^{\infty} g^i \in L^1(I).$$

We have

$$f_0 = \frac{1}{1-g} \quad \text{almost everywhere}$$

so

$$g = \frac{f_0 - 1}{f_0} \quad \text{almost everywhere}$$

and

$$J(f) = I(f f_0)$$

for $f \geq 0$, $f \in L^1(J)$.

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for $f \geq 0$, $f \in L^1(J)$. By breaking any $f \in L^1(J)$ into the difference of its positive and negative parts, we conclude that (3) holds for all $f \in L^1(J)$. The uniqueness of f_0 (almost everywhere (I)) follows from the uniqueness of g in $L^2(K)$. QED

Lemma 6.1 *The set where $g \geq 1$ has I measure zero.*

The Radon Nikodym theorem can be extended in two directions. First of all, let us continue with our assumption that I and J are bounded, but drop the absolute continuity requirement. Let us say that an integral H is **absolutely singular** with respect to I if there is a set N of I-measure zero such that $J(h) = 0$ for any h vanishing on N.

Let us now go back to Lemma 6.1. Define J_{sing} by

$$J_{sing}(f) = J(\mathbf{1}_N f).$$

Then J_{sing} is singular with respect to I, and we can write

$$J = J_{cont} + J_{sing}$$

where

$$J_{cont} = J - J_{sing} = J(\mathbf{1}_{N^c} \cdot).$$

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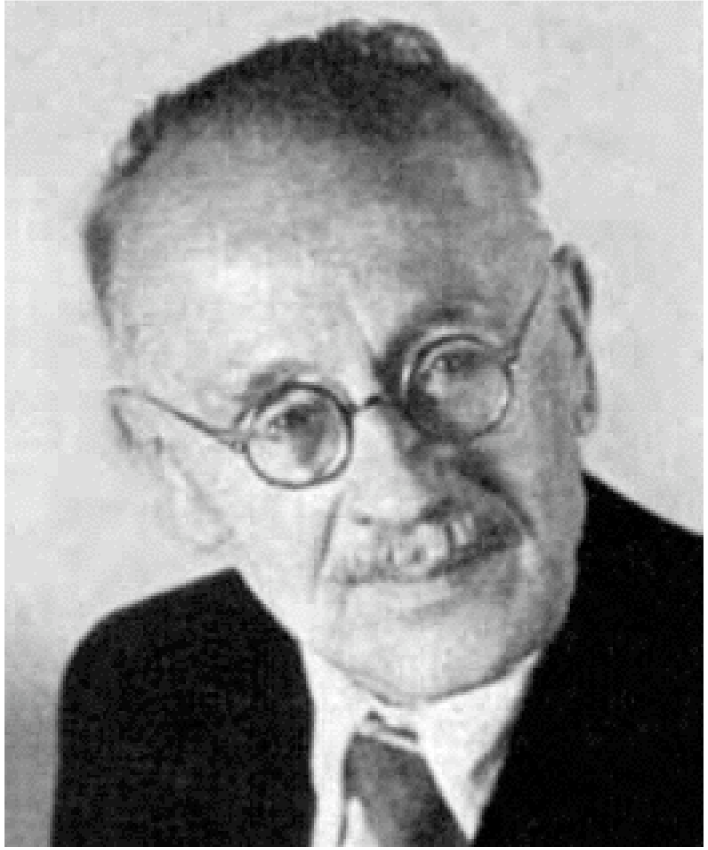
Then we can apply the rest of the proof of the Radon Nikodym theorem to J_{cont} to conclude that

$$J_{cont}(f) = I(f f_0)$$

where $f_0 = \sum_{i=1}^{\infty} (\mathbf{1}_{N^c} g)^i$ is an element of $L^1(I)$ as before. In particular, J_{cont} is absolutely continuous with respect to I.

A second extension is to certain situations where S is not of finite measure. We say that a function f is **locally** L^1 if $f\mathbf{1}_A \in L^1$ for every set A with $\mu(A) < \infty$. We say that S is **σ-finite** with respect to μ if S is a countable union of sets of finite μ measure. This is the same as saying that $\mathbf{1} = \mathbf{1}_S \in \mathcal{B}$. If S is σ-finite then it can be written as a disjoint union of sets of finite measure. If S is σ-finite with respect to both I and J it can be written as the disjoint union of countably many sets which are both I and J finite. So if J is absolutely continuous with respect I, we can apply the Radon-Nikodym theorem to each of these sets of finite measure, and conclude that there is an f_0 which is locally L^1 with respect to I, such that $J(f) = I(f f_0)$ for all $f \in L^1(J)$, and f_0 is unique up to almost everywhere equality.

Johann Radon



Born: 16 Dec 1887 in Tetschen, Bohemia (now Decin, Czech Republic)
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The dual space of L^p.

Recall that Hölder's inequality (1) says that

$$\left|\int fg d\mu\right| \le \|f\|_p \|g\|_q$$

if $f \in L^p$ and $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1.$

For the rest of this section we will assume without further mention that this relation between p and q holds. Hölder's inequality implies that we have a map from

$$L^q \to (L^p)^*$$

sending $g \in L^q$ to the continuous linear function on L^p which sends

$$f \mapsto I(fg) = \int fg d\mu.$$

Furthermore, Hölder's inequality says that the norm of this map from $L^q \to (L^p)^*$ is ≤ 1. In particular, this map is injective.

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The theorem we want to prove is that under suitable conditions on S and I (which are more general even that σ-finiteness) this map is surjective for $1 \leq p < \infty$.

We will first prove the theorem in the case where $\mu(S) < \infty$, that is when I is a bounded integral. For this we will will need a lemma:

The variations of a bounded functional.

Suppose we start with an arbitrary L and I. For each $1 \leq p \leq \infty$ we have the norm $\|\cdot\|_p$ on L which makes L into a real normed linear space. Let F be a linear function on L which is bounded with respect to this norm, so that

$$|F(f)| \leq C\|f\|_p$$

for all $f \in L$ where C is some non-negative constant. The least upper bound of the set of C which work is called $\|F\|_p$ as usual. If $f \geq 0 \in L$, define

$$F^+(f) := \text{lub}\{F(g) : 0 \leq g \leq f, \quad g \in L\}.$$

Then

$$F^+(f) \geq 0$$

$F^+(f) := \text{lub}\{F(g) : 0 \le g \le f, \quad g \in L\}$. Then $F^+(f) \ge 0$

and

$$F^+(f) \le \|F\|_p \|f\|_p$$

since $F(g) \le |F(g)| \le \|F\|_p \|g\|_p \le \|F\|_p \|f\|_p$ for all $0 \le g \le f$, $g \in L$, since $0 \le g \le f$ implies $|g|^p \le |f|^p$ for $1 \le p < \infty$ and also implies $\|g\|_\infty \le \|f\|_\infty$. Also

$$F^+(cf) = cF^+(f) \quad \forall\ c \ge 0$$

as follows directly from the definition. Suppose that f_1 and f_2 are both non-negative elements of L. If $g_1,\ g_2 \in L$ with

$$0 \le g_1 \le f_1 \ \text{ and } 0 \le g_2 \le f_2$$

$$F^+(f_1+f_2) \ge \text{lub}\, F(g_1+g_1) = \text{lub}\, F(g_1) + \text{lub}\, F(g_2)$$
$$= F^+(f_1) + F^+(f_2).$$

$$F^+(f_1+f_2) \geq \operatorname{lub} F(g_1+g_1) = \operatorname{lub} F(g_1)+\operatorname{lub} F(g_2) = F^+(f_1)+F^+(f_2).$$

On the other hand, if $g \in L$ satisfies $0 \leq g \leq (f_1 + f_2)$ then $0 \leq g \wedge f_1 \leq f_1$, and $g \wedge f_1 \in L$. Also $g - g \wedge f_1 \in L$ and vanishes at points x where $g(x) \leq f_1(x)$ while at points where $g(x) > f_1(x)$ we have $g(x) - g \wedge f_1(x) = g(x) - f_1(x) \leq f_2(x)$. So

$$g - g \wedge f_1 \leq f_2$$

and so

$$F^+(f_1+f_2) = \operatorname{lub} F(g) \leq \operatorname{lub} F(g \wedge f_1)+\operatorname{lub} F(g-g \wedge f_1) \leq F^+(f_1)+F^+(f_2).$$

So

$$F^+(f_1 + f_2) = F^+(f_1) + F^+(f_2)$$

if both f_1 and f_2 are non-negative elements of L.

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if both f_1 and f_2 are non-negative elements of L. Now write any $f \in L$ as $f = f_1 - g_1$ where f_1 and g_1 are non-negative. (For example we could take $f_1 = f^+$ and $g_1 = f^-$.) Define

$$F^+(f) = F^+(f_1) - F^+(g_1).$$

This is well defined, for if we also had $f = f_2 - g_2$ then $f_1 + g_2 = f_2 + g_1$ so

$$F^+(f_1)+F^+(g_2) = F^+(f_1+g_2) = F^+(f_2+g_1) = F^+(f_2)+F^+(g_1)$$

so

$$F^+(f_1) - F^+(g_1) = F^+(f_2) - F^+(g_2).$$

From this it follows that F^+ so extended is linear, and

$$|F^+(f)| \le F^+(|f|) \le \|F\|_p \|f\|_p$$

so F^+ is bounded.

Define F^- by

$$F^-(f) := F^+(f) - F(f).$$

As F^- is the difference of two linear functions it is linear. Since by its definition, $F^+(f) \geq F(f)$ if $f \geq 0$, we see that $F^-(f) \geq 0$ if $f \geq 0$. Clearly $\|F^-\| \leq \|F^+\|_p + \|F\| \leq 2\|F\|_p$. We have proved:

Proposition 7.1 *Every linear function on L which is bounded with respect to the $\|\cdot\|_p$ norm can be written as the difference $F = F^+ - F^-$ of two linear functions which are bounded and take non-negative values on non-negative functions.*

In fact, we could formulate this proposition more abstractly as dealing with a normed vector space which has an order relation consistent with its metric but we shall refrain from this more abstract formulation.

Duality of L^p and L^q when $\mu(S) < \infty$.

Theorem 7.1 *Suppose that $\mu(S) < \infty$ and that F is a bounded linear function on L^p with $1 \leq p < \infty$. Then there exists a unique $g \in L^q$ such that*

$$F(f) = (f, g) = I(fg).$$

Here $q = p/(p-1)$ if $p > 1$ and $q = \infty$ if $p = 1$.

Proof. Consider the restriction of F to L. We know that $F = F^+ - F^-$ where both F^+ and F^- are linear and non-negative and are bounded with respect to the $\|\cdot\|_p$ norm on L. The monotone convergence theorem implies that if $f_n \searrow 0$ then $\|f_n\|_p \to 0$ and the boundedness of F^+ with respect to the $\|\cdot\|_p$ says that

$$\|f_n\|_p \to 0 \quad \Rightarrow \quad F^+(f_n) \to 0.$$

So F^+ satisfies all the axioms for an integral, and so does F^-.

So F^+ satisfies all the axioms for an integral, and so does F^-. If f vanishes outside a set of I measure zero, then $\|f\|_p = 0$. Applied to a function of the form $f = \mathbf{1}_A$ we conclude that if A has $\mu = \mu_I$ measure zero, then A has measure zero with respect to the measures determined by F^+ or F^-. We can apply the Radon-Nikodym theorem to conclude that there are functions g^+ and g^- which belong to $L^1(I)$ and such that

$$F^{\pm}(f) = I(fg^{\pm})$$

for every f which belongs to $L^1(F^{\pm})$. In particular, if we set $g := g^+ - g^-$ then

$$F(f) = I(fg)$$

for every function f which is integrable with respect to both F^+ and F^-, in particular for any $f \in L^p(I)$. We must show that $g \in L^q$.

We first treat the case where $p > 1$. Suppose that $0 \leq f \leq |g|$ and that f is bounded. Then

$$I(f^q) \leq I(f^{q-1} \cdot \operatorname{sgn}(g)g) = F(f^{q-1} \cdot \operatorname{sgn}(g)) \leq \|F\|_p \|f^{q-1}\|_p.$$

So

$$I(f^q) \leq \|F\|_p (I(f^{(q-1)p}))^{\frac{1}{p}}.$$

Now $(q-1)p = q$ so we have

$$I(f^q) \leq \|F\|_p I(f^q)^{\frac{1}{p}} = \|F\|_q I(f^q)^{1-\frac{1}{q}}.$$

This gives

$$\|f\|_q \leq \|F\|_p$$

for all $0 \leq f \leq |g|$ with f bounded. We can choose such functions f_n with $f_n \nearrow |g|$. It follows from the monotone convergence theorem that $|g|$ and hence $g \in L^q(I)$. This proves the theorem for $p > 1$.

Let us now give the argument for $p = 1$. We want to show that $\|g\|_\infty \le \|F\|_1$. Suppose that $\|g\|_\infty \ge \|F\|_1 + \epsilon$ where $\epsilon > 0$. Consider the function $\mathbf{1}_A$ where

$$A := \{x : |g(x) \ge \|F\|_1 + \frac{\epsilon}{2}\}.$$

Then

$$(\|F\|_1 + \frac{\epsilon}{2})\mu(A) \le I(\mathbf{1}_A|g|) = I(\mathbf{1}_A \operatorname{sgn}(g)g) = F(\mathbf{1}_A \operatorname{sgn}(g))$$

$$\le \|F\|_1 \|\mathbf{1}_A \operatorname{sgn}(g)\|_1 = \|F\|_1 \mu(A)$$

which is impossible unless $\mu(A) = 0$, contrary to our assumption. QED

The case where $\mu(S) = \infty$.

Here the cases $p > 1$ and $p = 1$ may be different, depending on "how infinite S is".

Let us first consider the case where $p > 1$. If we restrict the functional F to any subspace of L^p its norm can only decrease. Consider a subspace consisting of all functions which vanish outside a subset S_1 where $\mu(S_1) < \infty$. We get a corresponding function g_1 defined on S_1 (and set equal to zero off S_1) with $\|g_1\|_q \leq \|F\|_p$ and $F(f) = I(fg_1)$ for all f belonging to this subspace. If (S_2, g_2) is a second such pair, then the uniqueness part of the theorem shows that $g_1 = g_2$ almost everywhere on $S_1 \cap S_2$. Thus we can consistently define g_{12} on $S_1 \cup S_2$. Let

$$b := \mathrm{lub}\{\|g_\alpha\|_q\}$$

taken over all such g_α. Since this set of numbers is bounded by $\|F\|_p$ this least upper bound is finite. We can therefore find a nested sequence of sets S_n and corresponding functions g_n such that

$$\|g_n\|_q \nearrow b.$$

$$\|g_n\|_q \nearrow b.$$

By the triangle inequality, if $n > m$ then

$$\|g_n - g_m\|_q \leq \|g_n\|_q - \|g_m\|_q$$

and so, as in your proof of the L^2 Martingale convergence theorem, this sequence is Cauchy in the $\|\cdot\|_q$ norm. Hence there is a limit $g \in L^q$ and g is supported on

$$S_0 := \bigcup S_n.$$

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$$S_0 := \bigcup S_n.$$

There can be no pair (S', g') with S disjoint from S_0 and $g' \neq 0$ on a subset of positive measure of S'. Indeed, if this were the case, then we could consider $g + g'$ on $S \cup S'$ and this would have a strictly larger $\| \cdot \|_q$ norm than $\|g\|_q = b$, contradicting the definition of b. (It is at this point in the argument that we use $q < \infty$ which is the same as $p > 1$.) Thus F vanishes on any function which is supported outside S_0. We have thus reduced the theorem to the case where S is σ-finite.

If S is σ-finite, decompose S into a disjoint union of sets A_i of finite measure. Let f_m denote the restriction of $f \in L^p$ to A_m and let h_m denote the restriction of g to A_m. Then

$$\sum_{m=1}^{\infty} f_m = f$$

as a convergent series in L^p and so

$$F(f) = \sum_m F(f_m) = \sum_m \int_{A_m} f_m h_m$$

and this last series converges to $I(fg)$ in L^1.

So we have proved that $(L^p)^* = L^q$ in complete generality when $p > 1$, and for σ-finite S when $p = 1$.

It may happen (and will happen when we consider the Haar integral on the most general locally compact group) that we don't even have σ-finiteness. But we will have the following more complicated condition: Recall that a set A is called **integrable** (by Loomis) if $\mathbf{1}_A \in \mathcal{B}$. Now suppose that

$$S = \bigcup_{\alpha} S_{\alpha}$$

where this union is disjoint, but possibly uncountable, of integrable sets, and with the property that every integrable set is contained in at most a countable union of the S_{α}. A set A is called **measurable** if the intersections $A \cap S_{\alpha}$ are all integrable, and a function is called **measurable** if its restriction to each S_{α} has the property that the restriction of f to each S_{α} belongs to $\mathcal{B}$, and further, that either the restriction of f^+ to every S_{α} or the restriction of f^- to every S_{α} belongs to L^1.

If we find ourselves in this situation, then we can find a g_α on each S_α since S_α is σ-finite, and piece these all together to get a g defined on all of S. If $f \in L^1$ then the set where $f \neq 0$ can have intersections with positive measure with only countably many of the S_α and so we can apply the result for the σ-finite case for $p = 1$ to this more general case as well.

Integration on locally compact Hausdorff spaces.

Suppose that S is a locally compact Hausdorff space. As in the case of $\mathbf{R}^n$, we can (and will) take L to be the space of continuous functions of compact support. Dini's lemma then says that if $f_n \in L \searrow 0$ then $f_n \to 0$ in the uniform topology.

If A is any subset of S we will denote the set of $f \in L$ whose support is contained in A by L_A.

Lemma 8.1 *A non-negative linear function I is bounded in the uniform norm on L_C whenever C is compact.*

Proof. Choose $g \geq 0 \in L$ so that $g(x) \geq 1$ for $x \in C$. If $f \in L_C$ then

$$|f| \leq \|f\|_\infty g$$

so

$$|I(f)| \leq I(|f|) \leq I(g) \cdot \|f\|_\infty. \quad \text{QED.}$$

Riesz representation theorems.

This is the same Riesz, but two more theorems.

Theorem 8.1 *Every non-negative linear functional I on L is an integral.*

Proof. This is Dini's lemma together with the preceding lemma. Indeed, by Dini we know that $f_n \in L \searrow 0$ implies that $\|f_n\|_\infty \searrow 0$. Since f_1 has compact support, let C be its support, a compact set. All the succeeding f_n are then also supported in C and so by the preceding lemma $I(f_n) \searrow 0$. QED

We have proved:

Proposition 7.1 *Every linear function on L which is bounded with respect to the $\|\cdot\|_p$ norm can be written as the difference $F = F^+ - F^-$ of two linear functions which are bounded and take non-negative values on non-negative functions.*

Theorem 8.2 *Let F be a bounded linear function on L (with respect to the uniform norm). Then there are two integrals I^+ and I^- such that*

$$F(f) = I^+(f) - I^-(f).$$

Proof. We apply Proposition 7.1 to the case of our L and with the uniform norm, $\|\cdot\|_\infty$. We get

$$F = F^+ - F^-$$

and an examination of he proof will show that in fact

$$\|F^\pm\|_\infty \leq \|F\|_\infty.$$

By the preceding theorem, $F^\pm$ are both integrals. QED

Fubini's theorem redux.

Theorem 8.3 *Let S_1 and S_2 be locally compact Hausdorff spaces and let I and J be non-negative linear functionals on $L(S_1)$ and $L(S_2)$ respectively. Then*

$$I_x(J_y h(x,y)) = J_y(I_x(h(x,y))$$

for every $h \in L(S_1 \times S_2)$ in the obvious notation, and this common value is an integral on $L(S_1 \times S_2)$.

Proof via Stone-Weierstrass. The equation in the theorem is clearly true if $h(x,y) = f(x)g(y)$ where $f \in L(S_1)$ and $g \in L(S_2)$ and so it is true for any h which can be written as a finite sum of such functions. Let h be a general element of $L(S_1 \times S_2)$. then we can find compact subsets $C_1 \subset S_1$ and $C_2 \subset S_2$ such that h is supported in the compact set $C_1 \times C_2$. The functions of the form

$$\sum f_i(x)g_i(y)$$

where the f_i are all supported in C_1 and the g_i in C_2, and the sum is finite, form an algebra which separates points. So for any $\epsilon > 0$ we can find a k of the above form with

$$\|h - k\|_\infty < \epsilon.$$

The functions of the form $\sum f_i(x)g_i(y)$ where the f_i are all supported in C_1 and the g_i in C_2, and the sum is finite, form an algebra which separates points. So for any $\epsilon > 0$ we can find a k of the above form with

$$\|h - k\|_\infty < \epsilon.$$

Recall:

Lemma 8.1 *A non-negative linear function I is bounded in the uniform norm on L_C whenever C is compact.*

Let B_1 and B_2 be bounds for I on $L(C_1)$ and J on $L(C_2)$ as provided by Lemma 7. Then

$$|J_y h(x, y) - \sum J(g_i) f_i(x)| = |[J_y(h - k)](x)| < \epsilon B_2.$$

$$|J_y h(x,y) - \sum J(g_i) f_i(x)| = |[J_y(h-k)](x)| < \epsilon B_2.$$

This shows that $J_y h(x,y)$ is the uniform limit of continuous functions supported in C_1 and so $J_y h(x,y)$ is itself continuous and supported in C_1. It then follows that $I_x(J_y(h)$ is defined, and that

$$|I_x(J_y h(x,y) - \sum I(f)_i J(g_i)| \le \epsilon B_1 B_2.$$

Doing things in the reverse order shows that

$$I_x(J_y h(x,y)) - J_y(I_x(h(x,y))| \le 2\epsilon B_1 B_2.$$

Since ϵ is arbitrary, this gives the equality in the theorem. Since this (same) functional is non-negative, it is an integral by the first of the Riesz representation theorems above. QED

A more precise version of the Riesz representation theorem.

It is useful to have a better description of the measure provided by the Riesz representation theorem just proved. In particular it is useful to know that

we can find a μ (which is possibly an extension of the μ given by our previous proof of the Riesz representation theorem) which is defined on a σ-field which contains the Borel field $\mathcal{B}(X)$. Recall that $\mathcal{B}(X)$ is the smallest σ-field which contains the open sets.

Let $\mathcal{F}$ be a σ-field which contains $\mathcal{B}(X)$. A (non-negative valued) measure μ on $\mathcal{F}$ is called **regular** if

1. $\mu(K) < \infty$ for any compact subset $K \subset X$.

2. For any $A \in \mathcal{F}$

$$\mu(A) = \inf\{\mu(U) : \ A \subset U, \ U \text{ open}\}$$

3. If $U \subset X$ is open then

$$\mu(U) = \sup\{\mu(K) : K \subset U, \quad K \text{ compact }\}.$$

The second condition is called **outer regularity** and the third condition is called **inner regularity**.

Here is the improved version of the Riesz representation theorem:

Theorem 14 *Let X be a locally compact Hausdorff space, L the space of continuous functions of compact support on X, and I a non-negative linear functional on L. Then there exists a σ-field $\mathcal{F}$ containing $\mathcal{B}(X)$ and a non-negative regular measure μ on $\mathcal{F}$ such that*

$$I(f) = \int f d\mu \tag{4}$$

for all $f \in L$. Furthermore, the restriction of μ to $\mathcal{B}(X)$ is unique.

Proof next time.