Math 212 Lecture 15

A more refined version of the Riesz representation theorem for measures on Hausdorff spaces.

I want to give an alternative proof of the Riesz representation theorem which will give some information about the possible σ-fields on which μ is defined. In particular, I want to show that we can find a μ (which is possibly an extension of the μ given by our previous proof of the Riesz representation theorem) which is defined on a σ-field which contains the Borel field $\mathcal{B}(X)$. Recall that $\mathcal{B}(X)$ is the smallest σ-field which contains the open sets.

Regular measures.

Let $\mathcal{F}$ be a σ-field which contains $\mathcal{B}(X)$. A (non-negative valued) measure μ on $\mathcal{F}$ is called **regular** if

1. $\mu(K) < \infty$ for any compact subset $K \subset X$.
2. For any $A \in \mathcal{F}$
$$\mu(A) = \inf\{\mu(U) : \ A \subset U, \ U \text{ open}\}$$
3. If $U \subset X$ is open then
$$\mu(U) = \sup\{\mu(K) : K \subset U, \quad K \text{ compact } \}.$$

The second condition is called **outer regularity** and the third condition is called **inner regularity**.

Statement of the theorem.

Here is the improved version of the Riesz representation theorem:

Theorem 14 *Let X be a locally compact Hausdorff space, L the space of continuous functions of compact support on X, and I a non-negative linear functional on L. Then there exists a σ-field $\mathcal{F}$ containing $\mathcal{B}(X)$ and a non-negative regular measure μ on $\mathcal{F}$ such that*

$$I(f) = \int f d\mu \tag{4}$$

for all $f \in L$. Furthermore, the restriction of μ to $\mathcal{B}(X)$ is unique.

Goal and method.

The proof of this theorem hinges on some topological facts whose true place is in the chapter on metric spaces, but I will prove them here. The importance of the theorem is that it will allow us to derive some conclusions about spaces which are very huge (such as the space of "all" paths in $\mathbf{R}^n$) but are nevertheless locally compact (in fact compact) Hausdorff spaces. It is because we want to consider such spaces, that the earlier proof, which hinged on taking limits of *sequences* in the very definition of the Daniell integral, is insufficient to get at the results we want.

Propositions in the topology of locally compact Hausdorff spaces.

Proposition 4 *Let X be a Hausdorff space, and let H and K be disjoint compact subsets of X. Then there exist disjoint open subsets U and V of X such that $H \subset U$ and $K \subset V$.*

This we actually did prove in the chapter on metric spaces.

Proposition 5 *Let X be a locally compact Hausdorff space, $x \in X$, and U an open set containing x. Then there exists an open set O such that*

- $x \in O$
- $\overline{O}$ *is compact, and*
- $\overline{O} \subset U$.

Proof. Choose an open neighborhood W of x whose closure is compact, which is possible since we are assuming that X is locally compact. Let $Z = U \cap W$ so that $\overline{Z}$ is compact and hence so is $H := \overline{Z} \setminus Z$. Take $K := \{x\}$ in the preceding proposition. We then get an open set V containing x which is disjoint from an open set G containing $\overline{Z} \setminus Z$. Take $O := V \cap Z$. Then $x \in O$ and $\overline{O} \subset \overline{Z}$ is compact and O has empty intersection with $\overline{Z} \setminus Z$, and hence is is contained in $Z \subset U$. QED

Proposition 6 *Let X be a locally compact Hausdorff space, $K \subset U$ with K compact and U open subsets of X. Then there exists a V with*

$$K \subset V \subset \overline{V} \subset U$$

with V open and $\overline{V}$ compact.

Proof. Each $x \in K$ has a neighborhood O with compact closure contained in U, by the preceding proposition. The set of these O cover K, so a finite subcollection of them cover K and the union of this finite subcollection gives the desired V.

Proposition 7 *Let X be a locally compact Hausdorff space, $K \subset U$ with K compact and U open. Then there exists a continuous function h with compact support such that*

$$\mathbf{1}_K \leq h \leq \mathbf{1}_U$$

and

$$\operatorname{supp}(h) \subset U.$$

Proof. Choose V as in Proposition 6. By Urysohn's lemma applied to the compact space $\overline{V}$ we can find a function $h : \overline{V} \to [0,1]$ such that $h = 1$ on K and $f = 0$ on $\overline{V} \setminus V$. Extend h to be zero on the complement of $\overline{V}$. Then h does the trick.

Proposition 8 *Let X be a locally compact Hausdorff space, $f \in L$, i.e. f is a continuous function of compact support on X. Suppose that there are open subsets $U_1, \ldots U_n$ such that*

$$\operatorname{supp}(f) \subset \bigcup_{i=1}^{n} U_i.$$

Then there are $f_1, \ldots, f_n \in L$ such that

$$\operatorname{supp}(f_i) \subset U_i$$

and

$$f = f_1 + \cdots + f_n.$$

If f is non-negative, the f_i can be chosen so as to be non-negative.

Proof. By induction, it is enough to consider the case $n = 2$. Let $K := \text{supp}(f)$, so $K \subset U_1 \cup U_2$. Let

$$L_1 := K \setminus U_1, \qquad L_2 := K \setminus U_2.$$

So L_1 and L_2 are disjoint compact sets. By Proposition 4 we can find disjoint open sets V_1, V_2 with

$$L_1 \subset V_1, \quad L_2 \subset V_2.$$

Set

$$K_1 := K \setminus V_1, \;\; K_2 := K \setminus V_2.$$

Then K_1 and K_2 are compact, and

$$K = K_1 \cup K_2, \quad K_1 \subset U_1,\ K_2 \subset U_2.$$

Choose h_1 and h_2 as in Proposition 7. Then set

$$\phi_1 := h_1, \quad \phi_2 := h_2 - h_1 \wedge h_2.$$

Then K_1 and K_2 are compact, and

$$K = K_1 \cup K_2, \quad K_1 \subset U_1, \; K_2 \subset U_2.$$

Choose h_1 and h_2 as in Proposition 7. Then set

$$\phi_1 := h_1, \quad \phi_2 := h_2 - h_1 \wedge h_2.$$

Then $\mathrm{supp}(\phi_1) = \mathrm{supp}(h_1) \subset U_1$ by construction, and $\mathrm{supp}(\phi_2) \subset \mathrm{supp}(h_2) \subset U_2$, the ϕ_i take values in $[0, 1]$, and, if $x \in K = \mathrm{supp}(f)$

$$\phi_1(x) + \phi_2(x) = (h_1 \vee h_2)(x) = 1.$$

Then set

$$f_1 := \phi_1 f, \quad f_2 := \phi_2 f.$$

QED

Proof of the uniqueness of the μ restricted to $\mathcal{B}(X)$.

It is enough to prove that

$$\mu(U) = \sup\{I(f) : \ f \in L, \ 0 \le f \le \mathbf{1}_U\} \tag{5}$$

$$= \sup\{I(f) : \ f \in L, \ 0 \le f \le \mathbf{1}_U, \ \ \mathrm{supp}(f) \subset U\} \tag{6}$$

for any open set U, since either of these equations determines μ on any open set U and hence for the Borel field.

Since $f \le \mathbf{1}_U$ and both are measurable functions, it is clear that $\mu(U) = \int \mathbf{1}_U$ is at least as large as the expression on the right hand side of (5). This in turn is as least as large as the right hand side of (6) since the supremum in (6) is taken over as smaller set of functions that that of (5). So it is enough to prove that $\mu(U)$ is $\le$ the right hand side of (6).

Let $a < \mu(U)$. Interior regularity implies that we can find a compact set $K \subset U$ with

$$a < \mu(K).$$

Take the f provided by Proposition 7. Then $a < I(f)$, and so the right hand side of (6) is $\geq a$. Since a was any number $< \mu(U)$, we conclude that $\mu(U)$ is $\leq$ the right hand side of 6). QED

Plan for the proof of existence.

We will

- define a function m^* defined on all subsets,
- show that it is an outer measure,
- show that the set of measurable sets in the sense of Caratheodory include all the Borel sets, and that
- integration with respect to the associated measure μ assigns $I(f)$ to every $f \in L$.

Definition of the outer measure.

Define m^* on open sets by

$$m^*(U) = \sup\{I(f): \ f \in L, 0 \leq f \leq \mathbf{1}_U, \ \mathrm{supp}(f) \subset U\}. \tag{7}$$

Clearly, if $U \subset V$ are open subsets, $m^*(U) \leq m^*(V)$. Next define m^* on an arbitrary subset by

$$m^*(A) = \inf\{m^*(U): \ A \subset U, \ U \text{ open}\}. \tag{8}$$

Since U is contained in itself, this does not change the definition on open sets. It is clear that $m^*(\emptyset) = 0$ and that $A \subset B$ implies that $m^*(A) \leq m^*(B)$.

So to prove

that m^* is an outer measure we must prove countable subadditivity. We will first prove countable subadditivity on open sets, and then use the $\epsilon/2^n$ argument to conclude countable subadditivity on all sets:

Suppose $\{U_n\}$ is a sequence of open sets. We wish to prove that

$$m^*\left(\bigcup_n U_n\right) \leq \sum_n m^*(U_n). \tag{9}$$

Set

$$U := \bigcup_n U_n,$$

and suppose that

$$f \in L, 0 \leq f \leq \mathbf{1}_U,\ \mathrm{supp}(f) \subset U.$$

$$U := \bigcup_n U_n,$$

and suppose that

$$f \in L, 0 \le f \le \mathbf{1}_U, \ \mathrm{supp}(f) \subset U.$$

Since supp(f) is compact and contained in U, it is covered by finitely many of the U_i. In other words, there is some finite integer N such that

$$\mathrm{supp}(f) \subset \bigcup_{n=1}^{N} U_n.$$

By Proposition 8 we can write

$$f = f_1 + \cdots + f_N, \quad \mathrm{supp}(f_i) \subset U_i, \quad i = 1, \ldots, N.$$

Then

$$I(f) = \sum I(f_i) \le \sum m^*(U_i),$$

To prove: $$m^*\left(\bigcup_n U_n\right) \leq \sum_n m^*(U_n). \tag{9}$$

For open sets we may use the definition

$$m^*(U) = \sup\{I(f) : \; f \in L, 0 \leq f \leq \mathbf{1}_U, \; \mathrm{supp}(f) \subset U\}. \tag{7}$$

Then

$$I(f) = \sum I(f_i) \leq \sum m^*(U_i),$$

using the definition (7). Replacing the finite sum on the right hand side of this inequality by the infinite sum, and then taking the supremum over f proves (9), where we use the definition (7) once again.

Next let $\{A_n\}$ be any sequence of subsets of X. We wish to prove that

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n).$$

This is automatic if the right hand side is infinite. So assume that

$$\sum_n m^*(A_n) < \infty$$

and choose open sets $U_n \supset A_n$ so that

$$m^*(U_n) \leq m^*(A_n) + \frac{\epsilon}{2^n}.$$

Then $U := \bigcup U_n$ is an open set containing $A := \bigcup A_n$ and

$$m^*(A) \leq m^*(U) \leq \sum m^*(U)_i \leq \sum_n m^*(A_n) + \epsilon.$$

Since ϵ is arbitrary, we have proved countable subadditivity.

Measurability of the Borel sets.

Let $\mathcal{F}$ denote the collection of subsets which are measurable in the sense of Caratheodory for the outer measure m^*. We wish to prove that $\mathcal{F} \supset \mathcal{B}(X)$. Since $\mathcal{B}(X)$ is the σ-field generated by the open sets, it is enough to show that every open set is measurable in the sense of Caratheodory, i.e. that

$$m^*(A) \geq m^*(A \cap U) + m^*(A \cap U^c) \tag{10}$$

for any open set U and any set A with $m^*(A) < \infty$

We wish to prove that

$$m^*(A) \geq m^*(A \cap U) + m^*(A \cap U^c) \tag{10}$$

for any open set U and any set A with $m^*(A) < \infty$: If $\epsilon > 0$, choose an open set $V \supset A$ with

$$m^*(V) \leq m^*(A) + \epsilon$$

which is possible by the definition (8). We will show that

$$m^*(V) \geq m^*(V \cap U) + m^*(V \cap U^c) - 2\epsilon. \tag{11}$$

This will then imply that

$$m^*(A) \geq m^*(A \cap U) + m^*(A \cap U^c) - 3\epsilon$$

We wish to prove that

$$m^*(V) \geq m^*(V \cap U) + m^*(V \cap U^c) - 2\epsilon. \qquad (11)$$

Using the definition (7), we can find an $f_1 \in L$ such that

$$f_1 \leq \mathbf{1}_{V \cap U} \quad \text{and} \quad \operatorname{supp}(f_1) \subset V \cap U$$

with

$$I(f_1) \geq m^*(V \cap U) - \epsilon.$$

Let $K := \operatorname{supp}(f_1)$. Then $K \subset U$ and so $K^c \supset U^c$ and K^c is open. Hence $V \cap K^c$ is an open set and

$$V \cap K^c \supset V \cap U^c.$$

$$V \cap K^c \supset V \cap U^c.$$

Using the definition (7), we can find an $f_2 \in L$ such that

$$f_2 \leq \mathbf{1}_{V \cap K^c} \quad \text{and} \quad \operatorname{supp}(f_2) \subset V \cap K^c$$

with

$$I(f_2) \geq m^*(V \cap K^c) - \epsilon.$$

But $m^*(V \cap K^c) \geq m^*(V \cap U^c)$ since $V \cap K^c \supset V \cap U^c$. So

$$I(f_2) \geq m^*(V \cap U^c) - \epsilon.$$

So

$$f_1 + f_2 \leq \mathbf{1}_K + \mathbf{1}_{V \cap K^c} \leq \mathbf{1}_V$$

since $K = \operatorname{supp}(f_1) \subset V$ and $\operatorname{supp}(f_2) \subset V \cap K^c$.

Conclusion of the proof that Borel sets are measurable.

We wish to prove that

$$m^*(V) \geq m^*(V \cap U) + m^*(V \cap U^c) - 2\epsilon. \qquad (11)$$

We have

$$I(f_1) \geq m^*(V \cap U) - \epsilon. \qquad I(f_2) \geq m^*(V \cap K^c) - \epsilon.$$

$$\mathrm{supp}(f_1 + f_2) \subset (K \cup V \cap K^c) = V.$$

Thus $f = f_1 + f_2 \in L$ and so by (7),

$$I(f_1 + f_2) \leq m^*(V).$$

This proves (11) and hence that all Borel sets are measurable.

Compact sets have finite measure.

Let μ be the measure associated to m on the σ-field $\mathcal{F}$ of measurable sets. We will now prove that μ is regular. The condition of outer regularity is automatic, since this was how we defined $\mu(A) = m^*(A)$ for a general set.

If K is a compact subset of X, we can find an $f \in L$ such that $\mathbf{1}_K \leq f$ by Proposition 7. Let $0 < \epsilon < 1$ and set

$$U_\epsilon := \{x : \ f(x) > 1 - \epsilon\}.$$

Then U_ϵ is an open set containing K. If $0 \leq g \in L$ satisfies $g \leq \mathbf{1}_{U_\epsilon}$, then $g = 0$ on U_ϵ^c, and for $x \in U_\epsilon$, $g(x) \leq 1$ while $f(x) > 1 - \epsilon$. So

$$g \leq \frac{1}{1-\epsilon} f$$

and hence, by (7)

$$m^*(U_\epsilon) \leq \frac{1}{1-\epsilon} I(f).$$

So, by (8)

$$\mu(K) \leq m^*(U_\epsilon) \leq \frac{1}{1-\epsilon} I(f) < \infty.$$

Reviewing the preceding argument, we see that we have in fact proved the more general statement

Proposition 9 *If A is any subset of X and $f \in L$ is such that*

$$\mathbf{1}_A \leq f$$

then

$$m^*(A) \leq I(f).$$

Interior regularity.

We now prove interior regularity, which will be very important for us. We wish to prove that

$$\mu(U) = \sup\{\mu(K) : K \subset U, \quad K \text{ compact }\},$$

for any open set U, where, according to (7),

$$m^*(U) = \sup\{I(f) : \ f \in L, 0 \leq f \leq \mathbf{1}_U, \ \mathrm{supp}(f) \subset U\}.$$

Since $\mathrm{supp}(f)$ is compact, and contained in U, we will be done if we show that

$$f \in L, \quad 0 \leq f \leq 1 \Rightarrow I(f) \leq \mu(\mathrm{supp}(f)). \qquad (12)$$

We wish to prove that

$$f \in L, \quad 0 \le f \le 1 \Rightarrow I(f) \le \mu(\mathrm{supp}(f)). \qquad (12)$$

So let V be an open set containing $\mathrm{supp}(f)$. By definition (7),

$$\mu(V) \ge I(f)$$

and, since V is an arbitrary open set containing $\mathrm{supp}(f)$, we have

$$\mu(\mathrm{supp}(f)) \ge I(f)$$

using the definition (8) of $m^*(\mathrm{supp}(f))$.

In the course of this argument we have proved

Proposition 10 *If $g \in L, 0 \le g \le \mathbf{1}_K$ where K is compact, then*

$$I(g) \le \mu(K).$$

Conclusion of the proof.

Finally, we must show that all the elements of L are integrable with respect to μ and

$$I(f) = \int f d\mu. \tag{13}$$

Since the elements of L are continuous, they are Borel measurable. As every $f \in L$ can be written as the difference of two non-negative elements of L, and as both sides of (13) are linear in f, it is enough to prove (13) for non-negative functions.

Following Lebesgue, divide the "y-axis" up into intervals of size ϵ. That is, let ϵ be a positive number, and, for every positive integer n set

$$f_n(x) := \begin{cases} 0 & \text{if} \quad f(x) \leq (n-1)\epsilon \\ f(x) - (n-1)\epsilon & \text{if} \quad (n-1)\epsilon < f(x) \leq n\epsilon \\ \epsilon & \text{if} \quad n\epsilon < f(x) \end{cases}$$

If $(n-1)\epsilon \geq \|f\|_\infty$ only the first alternative can occur, so all but finitely many of the f_n vanish, and they all are continuous and have compact support so belong to L. Also

$$f = \sum f_n$$

this sum being finite, as we have observed, and so

$$I(f) = \sum I(f_n).$$

Set $K_0 := \text{supp}(f)$ and

$$K_n := \{x : f(x) \geq n\epsilon\} \quad n = 1, 2, \ldots.$$

Then the K_i are a nested decreasing collection of compact sets, and

$$\epsilon \mathbf{1}_{K_n} \leq f_n \leq \epsilon \mathbf{1}_{K_{n-1}}.$$

By Propositions 9 and10 we have

$$\epsilon\mu(K_n) \leq I(f_n) \leq \epsilon\mu(K_{n-1}).$$

On the other hand, the monotonicity of the integral (and its definition) imply that

$$\epsilon\mu(K_n) \leq \int f_n d\mu \leq \epsilon\mu(K_{n-1}).$$

Summing these inequalities gives

$$
\begin{aligned}
\epsilon \sum_{i=1}^{N} \mu(K_n) &\quad \leq I(f) &\quad \leq \epsilon \sum_{i=0}^{N-1} \mu(K_n) \\
\epsilon \sum_{i=1}^{N} \mu(K_n) &\quad \leq \int f d\mu &\quad \epsilon \sum_{i=0}^{N-1} \mu(K_n)
\end{aligned}
$$

where N is sufficiently large. Thus $I(f)$ and $\int f d\mu$ lie within a distance

$$
\epsilon \sum_{i=0}^{N-1} \mu(K_n) - \epsilon \sum_{i=1}^{N} \mu(K_n) = \epsilon\mu(K_0) - \epsilon\mu(K_N) \leq \epsilon\mu(\text{supp}(f))
$$

of one another. Since ϵ is arbitrary, we have proved (13) and completed the proof of the Riesz representation theorem.