1. Suppose $u \in W^{2,p}(U) \cap W_0^{1,p}(U)$. For $2 \le p < \infty$, is it correct that

$$\int |\nabla u|^p dx \le C \left(\int |u|^p \right)^{1/2} \left(\int |\nabla^2 u|^p \right)^{1/2} ?$$

Prove your answer.

2. Let K be the kernel

$$K(x,y) = |x|^{-1}|x-y|^{-1}|y|^{-1}$$

where $x, y \in \mathbb{R}^3$. Is K a bounded operator in $L^2(\mathbb{R}^3)$? Prove your answer. K is a bounded operator in $L^2(\mathbb{R}^3)$ iff

$$\left| \int\int f(x)f(y)K(x,y)dxdy \right| \le \|f\|_2^2$$

Hint: Try to follow the proof of $\| |x|^{-\alpha} I_\alpha f \|_p \le C \|f\|_p$ given in the class. Here

$$I_\alpha f(x) = \int |x-y|^{-n+\alpha} f(y) dy$$

3. i) Suppose $1 \le p_0 < p < p_1$ and $f \in L_w^p$. Prove that there is a constant C such that for any $M > 0$, we can write $f = g_M + h_M$ and

$$\|g_M\|_{p_0} \le \kappa M^{1-(p/p_0)}, \quad \|h_M\|_{p_1} \le \kappa M^{1-(p/p_1)}$$

ii) Prove the smallest κ and $\|f\|_{L_w^p}$ are equivalent, i.e, there is a constant C such that $C^{-1}\kappa \le \|f\|_{L_w^p} \le C\kappa$. Hint: Suppose f is positive. Then

$$\{x : f(x) > M\} \subset \{x : g_M(x) > M/2\} \cup \{x : h_M(x) > M/2\}$$

iii) Prove the Hunt's interpolation theorem. Hunt's theorem is stated in Reed-Simon Vol II, page 31. The definition of weak L_p norm is given by (3) in page 106 of Analysis.

4. Suppose μ is a probability measure and $f \ge 0$ with $\int f d\mu = 1$. Prove that

$$S(f) := \int f \log f d\mu \ge 0 \tag{i}$$

For any $\beta > 0$ and g real:

$$\int fg \le \beta^{-1} \log \int e^{\beta g} d\mu + \beta^{-1} S(f) \tag{ii}$$

Suppose $\|g\|_\infty \le 1$. Then for any $\beta > 0$:

$$\beta^{-1} \log \int e^{\beta g} d\mu \le \int g d\mu + \beta/2 \tag{iii}$$

Prove that

$$\left[\int |f-1| d\mu \right]^2 \le C \int f \log f d\mu \tag{iv}$$

(iv) is hard, but the proof is not long. The proof of (iv) in this approach uses (ii) and (iii). If you can prove (iv) directly, you don't have to do (i)-(iii).