1. Suppose $u(x,t)$ is a smooth function with compact support in $\mathbb{R}^n$ for each $t \in I = [0,1]$. Prove that

$$\int_I \int_{\mathbb{R}^n} |u|^{2(n+2)/n} dxdt \le C \sup_{t\in I} \left(\int |u(x,t)|^2 \right)^{2/n} \int_I \int_{\mathbb{R}^n} |\nabla u|^2 dxdt$$

2. Suppose μ, ν are two probability measures on $\mathbb{R}^n$ and they satisfy the l.s.i. in the sense that for $\omega = \mu$ or ν and $f \ge 0$ with $\int f\omega = 1$ the following inequality holds:

$$\int f \log f d\omega \le C \int |\nabla \sqrt{f}|^2 d\omega. \tag{l.s.i.}$$

Prove that l.s.i. holds for $\mu \otimes \nu$, i.e., for $f \ge 0$ with $\int f d\mu d\nu = 1$:

$$\int f(x,y) \log f(x,y) d\mu(x) d\nu(y) \le \int \left[|\nabla_x + \sqrt{f}|^2 |\nabla_y \sqrt{f}|^2 \right] d\mu(x) d\nu(y).$$

Hint: Let $\bar{f}(x) = \int f(x,y) d\nu(y)$ and write

$$\int f(x,y) \log f(x,y) d\mu(x) d\nu(y)$$

$$= \left[\int f(x,y) \log f(x,y) d\mu(x) d\nu(y) - \int \bar{f}(x) \log \bar{f}(x) d\mu(x) \right] + \int \bar{f}(x) \log \bar{f}(x) d\mu(x)$$

Then use l.s.i. for ν and μ.

3. Suppose μ is a probability measure that (l.s.i.) holds. Prove that the Poincare inequality in the following sense holds as well, i.e., for $\int g d\mu = 0$:

$$\int g^2 d\mu \le C \int |\nabla g|^2 d\mu$$

Hint: Substitute $u = (1 + \varepsilon g)$ in (l.s.i.) and let $\varepsilon \to 0$. Consider first the case $\|g\|_\infty < \infty$.

4. Suppose u is a smooth solution of the equation in $\mathbb{R}^n$:

$$\partial_t u = b\nabla u + \Delta u$$

where b is a smooth vector field satisfying the divergence free condition $\nabla \cdot b = 0$. Prove that

$$\|u(t)\|_{L^\infty} \le C t^{-n/2} \|u(t=0)\|_{L^1}$$

Here we assume that $\sup_{0\le s\le t} \sup_{1\le p\le\infty} \|u(s)\|_{L^p} < A$, but C is independent of A.

Hint: Mimic Nash's proof.