Math 212b Lecture 1

Review: The spectral theorem for unbounded operators via Gelfand.

For the first month of this semester I want to concentrate on the spectral theorem for unbounded self-adjoint operators and on applications of this theorem. We proved this theorem last semester via the Gelfand transform, and I will review the ingredients of this proof in today's lecture. I plan to give two more proofs of the spectral theorem and outline a fourth proof. We will study the meaning of the point spectrum (the eigenvalues) and the continuous spectrum, Stone's theorem, applications of the spectral theorem to some classical partial differential equations, the Rayleigh-Ritz principle for the discrete spectrum, some abstract scattering theory, and the Stone - von Neumann theorem about the canonical commutation relations of quantum mechanics.

Review: operators and their domains.

Let B and C be Banach spaces. We make $B \oplus C$ into a Banach space via

$$\|\{x, y\}\| = \|x\| + \|y\|.$$

Here we are using $\{x, y\}$ to denote the ordered pair of elements $x \in B$ and $y \in C$ so as to avoid any conflict with our notation for scalar product in a Hilbert space. So $\{x, y\}$ is just another way of writing $x \oplus y$.

A subspace

$$\Gamma \subset B \oplus C$$

will be called a **graph** (more precisely a graph of a linear transformation) if

$$\{0, y\} \in \Gamma \quad \Rightarrow \quad y = 0.$$

Another way of saying the same thing is

$$\{x, y_1\} \in \Gamma \quad \text{and} \quad \{x, y_2\} \in \Gamma \quad \Rightarrow \quad y_1 = y_2.$$

In other words, if $\{x, y\} \in \Gamma$ then y is determined by x. So let $D(\Gamma)$ denote the set of all $x \in B$ such that there is a $y \in C$ with $\{x, y\} \in \Gamma$. Then $D(\Gamma)$ is a linear subspace of B, but, and this is very important, $D(\Gamma)$ is *not* necessarily a closed subspace. We have a linear map

$$T(\Gamma): \ D(\Gamma) \to C, \quad Tx = y \quad \text{where} \quad \{x, y\} \in \Gamma.$$

Equally well, we could start with the linear transformation: Suppose we are given a (not necessarily closed) subspace $D(T) \subset B$ and a linear transformation

$$T : D(T) \to C.$$

We can then consider its graph $\Gamma(T) \subset B \oplus C$ which consists of all

$$\{x, Tx\}.$$

Review: Closed linear transformations.

A linear transformation is said to be **closed** if its graph is a closed subspace of $B \oplus C$. Let us disentangle what this says for the operator T. It says that if $f_n \in D(T)$ then

$$f_n \to f \text{ and } Tf_n \to g \quad \Rightarrow \quad f \in D(T) \text{ and } Tf = g.$$

This is a much weaker requirement than continuity. Continuity of T would say that $f_n \to f$ alone would imply that Tf_n converges to Tf. Closedness says that if we know that *both* f_n converges and $g_n = Tf_n$ converges then we can conclude that $f = \lim f_n$ lies in $D(T)$ and that $Tf = g$.

An important theorem, known as the *closed graph theorem* says that if T is closed and $D(T)$ is all of B then T is bounded. As we will not need to use this theorem in this lecture, we will not present its proof here.

Review: The adjoint.

Suppose that we have a linear operator $T : D(T) \to C$ and let us make the hypothesis that

$$D(T) \textit{ is dense in } B.$$

Any element of B^* is then completely determined by its restriction to $D(T)$. Now consider

$$\Gamma(T)^* \subset C^* \oplus B^*$$

defined by

$$\{\ell, m\} \in \Gamma(T)^* \quad \Leftrightarrow \quad \langle \ell, Tx \rangle = \langle m, x \rangle \quad \forall\ x \in D(T). \qquad (2)$$

Since m is determined by its restriction to $D(T)$, we see that $\Gamma^* = \Gamma(T^*)$ is indeed a graph. (It is easy to check that it is a linear subspace of $C^* \oplus B^*$.) In other words we have defined a linear transformation

$$T^* := T(\Gamma(T)^*)$$

$$T^* := T(\Gamma(T)^*)$$

whose domain consists of all $\ell \in C^*$ such that there exists an $m \in B^*$ for which $\langle \ell, Tx \rangle = \langle m, x \rangle \quad \forall\ x \in D(T)$.

If $\ell_n \to \ell$ and $m_n \to m$ then the definition of convergence in these spaces implies that for any $x \in D(T)$ we have

$$\langle \ell, Tx \rangle = \lim \langle \ell_n, Tx \rangle = \lim \langle m_n, x \rangle = \langle m, x \rangle.$$

If we let x range over all of $D(T)$ we conclude that Γ^* is a closed subspace of $C^* \oplus B^*$. In other words we have proved

Theorem 1 *If $T : D(T) \to C$ is a linear transformation whose domain $D(T)$ is dense in B, it has a well defined adjoint T^* whose graph is given by (2). Furthermore T^* is a closed operator.*

Self-adjoint operators.

Now let us restrict to the case where $B = C = H$ is a Hilbert space, so we may identify $B^* = C^* = H^*$ with H via the Riesz representation theorem. If $T : D(T) \to H$ is an operator with $D(T)$ dense in H we may identify the domain of T^* as consisting of all $\{g, h\} \in H \oplus H$ such that

$$(Tx, g) = (x, h) \quad \forall x \in D(T)$$

and then write

$$(Tx, g) = (x, T^*g) \quad \forall\ x \in D(T), \quad g \in D(T^*).$$

We now come to the central definition: An operator A defined on a domain $D(A) \subset H$ is called **self-adjoint** if

- $D(A)$ is dense in H,
- $D(A) = D(A^*)$, and
- $Ax = A^*x \; \forall x \in D(A)$.

The conditions about the domain $D(A)$ are rather subtle, and we shall go into some of their subtleties in a later lecture. For the moment we record one immediate consequence of the theorem of the preceding section:

Proposition 1 *Any self adjoint operator is closed.*

The resolvent.

The following theorem will be central for us.

Theorem 2 *Let A be a self-adjoint operator on a Hilbert space H with domain $D = D(A)$. Let*

$$c = \lambda + i\mu, \quad \mu \neq 0$$

be a complex number with non-zero imaginary part. Then

$$(cI - A) : D(A) \to H$$

is bijective. Furthermore the inverse transformation

$$(cI - A)^{-1} : H \to D(A)$$

is bounded and in fact

$$\|(cI - A)^{-1}\| \leq \frac{1}{|\mu|}. \tag{3}$$

Proof. Let $g \in D(A)$ and set

$$f = (cI - A)g = [\lambda I - A]g + i\mu g.$$

Then $\|f\|^2 = (f, f) =$

$$\|[cI - A]g\|^2 + \mu^2\|g\|^2 + ([\lambda I - A]g, i\mu g) + (i\mu g, [\lambda I - A]g).$$

I claim that these last two terms cancel. Indeed, since $g \in D(A)$ and A is self adjoint we have

$$(\mu g, [\lambda I - A]g) = (\mu[\lambda I - A]g, g) = ([\lambda I - A]g, \mu g)$$

since μ is real. Hence

$$([\lambda I - A]g, i\mu g) = -i(\mu g, [\lambda I - A]g).$$

We have thus proved that

$$\|f\|^2 = \|(\lambda I - A)g\|^2 + \mu^2\|g\|^2. \tag{4}$$

In particular

$$\|f\|^2 \geq \mu^2\|g\|^2$$

for all $g \in D(A)$. Since $|\mu| > 0$, we see that $f = 0 \Rightarrow g = 0$ so $(cI - A)$ is injective on $D(A)$, and furthermore that that $(cI - A)^{-1}$ (which is defined on im $(cI - A)$)satisfies (3). We must show that this image is all of H.

First we show that the image is dense. For this it is enough to show that there is no $h \neq 0 \in H$ which is orthogonal to im $(cI - A)$. So suppose that

$$([cI - A]g, h) = 0 \quad \forall g \in D(A).$$

Then

$$(g, \overline{c}h) = (cg, h) = (Ag, h) \;\; \forall g \in D(A)$$

which says that $h \in D(A^*)$ and $A^*h = \overline{c}h$. But A is self adjoint so $h \in D(A)$ and $Ah = \overline{c}h$. Thus

$$\overline{c}(h, h) = (\overline{c}h, h) = (Ah, h) = (h, Ah) = (h, \overline{c}h) = c(h, h).$$

Since $c \neq \overline{c}$ this is impossible unless $h = 0$. We have now established that the image of $cI - A$ is dense in H.

$$\|(cI-A)^{-1}\| \leq \frac{1}{|\mu|}. \tag{3}$$

We now prove that it is all of H. So let $f \in H$. We know that we can find

$$f_n = (cI-A)g_n, \quad g_n \in D(A) \quad \text{with } f_n \to f.$$

The sequence f_n is convergent, hence Cauchy, and from (3) applied to elements of $D(A)$ we know that

$$\|g_m - g_n\| \leq |\mu|^{-1}\|f_n - f_m\|.$$

Hence the sequence $\{g_n\}$ is Cauchy, so $g_n \to g$ for some $g \in H$. But we know that A is a closed operator. Hence $g \in D(A)$ and $(cI-A)g = f$. QED

The resolvent and the spectrum.

The operator

$$R_z = R_z(A) = (zI - A)^{-1}$$

is called the **resolvent** of A when it exists as a bounded operator. The set of $z \in \mathbf{C}$ for which the resolvent exists is called the **resolvent set** and the complement of the resolvent set is called the **spectrum** of the operator. The preceding theorem asserts that the spectrum of a self-adjoint operator is a subset of the real numbers.

The resolvent equation.

Let z and w both belong to the resolvent set. We have

$$wI - A = (w - z)I + (zI - A).$$

Multiplying this equation on the left by R_w gives

$$I = (w - z)R_w + R_w(zI - A),$$

and multiplying this on the right by R_z gives

$$R_z - R_w = (w - z)R_w R_z.$$

From this it follows (interchanging z and w) that $R_z R_w = R_w R_z$

From this it follows (interchanging z and w) that $R_z R_w = R_w R_z$, in other words all resolvents R_z commute with one another and we can also write the preceding equation as

$$R_z - R_w = (w - z)R_z R_w. \tag{5}$$

This equation, which is known as the **resolvent equation** dates back to the theory of integral equations in the nineteenth century.

It follows from the resolvent equation that $z \mapsto R_z$ (for fixed A) is a continuous function of z. Once we know that the resolvent is a continuous function of z, we may divide the resolvent equation by $(z - w)$ if $z \neq w$ and, if w is interior to the resolvent set, conclude that

$$\lim_{z \to w} \frac{R_z - R_w}{z - w} = -R_w^2.$$

This says that the "derivative in the complex sense" of the resolvent exists and is given by $-R_z^2$. In other words, the resolvent is a "holomorphic operator valued" function of z.

Proposition 2 *Let z belong to the resolvent set. The the open disk of radius $\|R_z\|^{-1}$ about z belongs to the resolvent set and on this disk we have*

$$R_w = R_z(I + (z-w)R_z + (z-w)^2 R_z^2 + \cdots). \tag{6}$$

Proof. The series on the right converges in the uniform topology since $|z-w| < \|R_z\|^{-1}$. Multiplying this series by $(zI - A) - (z-w)I$ gives I. But $zI - A - (z-w)I = wI - A$. So the right hand side is indeed R_w. QED

This suggests that we can develop a "Cauchy theory" of integration of functions such as the resolvent, and we shall do so, eventually leading to a proof of the spectral theorem for unbounded self-adjoint operators.

However we first give a proof (following the treatment in Reed-Simon) in which we derive the spectral theorem for unbounded operators from the Gelfand representation theorem applied to the closed algebra generated by the *bounded* normal operators $(\pm iI - A)^{-1}$.

The multiplication operator form of the spectral theorem.

We first state this theorem for closed commutative self-adjoint algebras of (bounded) operators. Recall that "self-adjoint" in this context means that if $T \in B$ then $T^* \in B$.

Theorem 3 *Let B be a commutative closed self-adjoint subalgebra of the algebra of all bounded operators on a separable Hilbert space H. Then there exists a measure space $(M, \mathcal{F}, \mu)$ with $\mu(M) < \infty$, a unitary isomorphism*

$$W : H \to L_2(M, \mu),$$

and a map

$$B \to \quad \text{bounded measurable functions on } M, \qquad T \mapsto \tilde{T}$$

such that

$$[(WTW^{-1})f](m) = \tilde{T}(m)f(m).$$

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such that

$$[(WTW^{-1})f](m) = \tilde{T}(m)f(m).$$

In fact, M can be taken to be a finite or countable disjoint union of $\mathcal{M} = \mathrm{Mspec}(B)$

$$M = \bigcup_{1}^{N} \mathcal{M}_i, \qquad \mathcal{M}_i = \mathcal{M}$$

$N \in \mathbf{Z}_+ \cup \infty$ and

$$\tilde{T}(m) = \hat{T}(m) \quad \textit{if} \;\; m \in \mathcal{M}_i = \mathcal{M}.$$

In short, the theorem says that any such B is isomorphic to an algebra of multiplication operators on an L_2 space. We prove the theorem in two stages.

Cyclic vectors.

An element $x \in H$ is called a **cyclic vector** for B if $Bx = H$. In more mundane terms this says that the space of linear combinations of the vectors Tx, $T \in B$ are dense in H.

For example, if B consists of all multiples of the identity operator, then Bx consists of all multiples of x, so B can not have a cyclic vector unless H is one dimensional. More generally, if H is finite dimensional and B is the algebra generated by a self-adjoint operator, then B can not have a cyclic vector if A has a repeated eigenvalue.

Proposition 3 *Suppose that x is a cyclic vector for B. Then it is a cyclic vector for the projection valued measure P on $\mathcal{M}$ associated to B in the sense that the linear combinations of the vectors $P(U)x$ are dense in H as U ranges over the Borel sets on $\mathcal{M}$.*

Proof. Suppose not. Then there exists a non-zero $y \in H$ such that

$$(P(U)x, y) = 0$$

for all Borel subset U of $\mathcal{M}$. Then

$$(Tx, y) = \int_M \hat{T} d(P(U)x, y) = 0$$

which contradicts the assumption that the linear combinations of the Tx are dense in H. QED

Let us continue with the assumption that x is a cyclic vector for B. Let

$$\mu = \mu_{x,x}$$

so

$$\mu(U) = (P(U)x, x).$$

This is a finite measure on $\mathcal{M}$, in fact

$$\mu(\mathcal{M}) = \|x\|^2. \tag{7}$$

We will construct a unitary isomorphism of H with $L_2(\mathcal{M}, \mu)$ starting with the assignment

$$Wx = \mathbf{1} = \mathbf{1}_{\mathcal{M}}.$$

We would like this to be a B morphism, even a morphism for the action of multiplication by bounded Borel functions. This forces the definition

$$WP(U)x = \mathbf{1}_U \mathbf{1} = \mathbf{1}_U.$$

This then forces

$$W[c_1 P(U_1)x + \cdots c_n P(U_n)x] = s$$

for any simple function

$$s = c_1 \mathbf{1}_{U_1} + \cdots c_n \mathbf{1}_{U_n}.$$

A direct check shows that this is well defined for simple functions.

We can write this map as $W\left[O(s)x\right] = s,$

and another direct check shows that $\|W\left[O(s)x\right]\| = \|s\|_2$

where the norm on the right is the L_2 norm relative to the measure μ. Since the simple functions are dense in $L_2(\mathcal{M}, \mu)$ and the vectors $O(s)x$ are dense in H this extends to a unitary isomorphism of H onto $L_2(M, \mu)$. Furthermore,

$$W^{-1}(f) = O(f)x$$

for any $f \in L_2(\mathcal{M}, \mu)$. For simple functions, and therefore for all $f \in L_2(\mathcal{M}, \mu)$ we have $W^{-1}(\hat{T}f) = O(\hat{T}f)x = TO(f)x = TW^{-1}(f)$

or

$$(WTW^{-1})f = \hat{T}f$$

which is the assertion of the theorem. In other words we have proved the theorem under the assumption of the existence of a cyclic vector.

The general case.

Start with any non-zero vector x_1 and consider $H_1 = Bx_1 =$ the closure of linear combinations of Tx_1, $T \in B$. The space H_1 is a closed subspace of H which is invariant under B, i.e. $TH_1 \subset H_1 \ \ \forall T \in B$. Therefore the space $H_1^{\perp}$ is also invariant under B since if $(x_1, y) = 0$ then

$$(x_1, Ty) = (T^* x_1, y) = 0 \ \text{ since } T^* \in B.$$

Now if $H_1 = H$ we are done, since x_1 is a cyclic vector for B acting on H_1. If not choose a non-zero $x_2 \in H_2$ and repeat the process. We can choose a collection of non-zero vectors z_i whose linear combinations are dense in H - this is the separability assumption. So we may choose our x_i to be obtained from orthogonal projections applied to the z_i. In other words we have

$$H = H_1 \oplus H_2 \oplus H_3 \oplus \cdots$$

where this is either a finite or a countable Hilbert space (completed) direct sum.

Let us also take care to choose our x_n so that

$$\sum \|x_n\|^2 < \infty$$

which we can do, since $c_n x_n$ is just as good as x_n for any $c_n \neq 0$. We have a unitary isomorphism of H_n with $L_2(\mathcal{M}, \mu_n)$ where $\mu_n(U) = (P(U)x_n, x_n)$. In particular,

$$\mu_n(\mathcal{M}) = \|x_n\|^2.$$

So if we take M to be the disjoint union of copies $\mathcal{M}_n$ of $\mathcal{M}$ each with measure μ_n then the total measure of M is finite and

$$L_2(M) = \bigoplus L_2(\mathcal{M}_n, \mu_n)$$

where this is either a finite direct sum or a (Hilbert space completion of) a countable direct sum. Thus the theorem for the cyclic case implies the theorem for the general case. QED

The spectral theorem for unbounded self-adjoint operators, multiplication operator form.

We now let A be a (possibly unbounded) self-adjoint operator, and we apply the previous theorem to the algebra generated by the bounded operators $(\pm iI - A)^{-1}$ which are the adjoints of one another. Observe that there is no non-zero vector $y \in H$ such that

$$(A + iI)^{-1} y = 0.$$

Indeed if such a $y \in H$ existed, we would have

$$0 = (x, (A+iI)^{-1}y) = ((A-iI)^{-1}x, y) = -((iI-A)^{-1}x, y) \quad \forall\, x \in H$$

and we know that the image of $(iI - A)^{-1}$ is $D(A)$ which is dense in H.

Plan.

Now consider the function $\left((A+iI)^{-1}\right)^{\sim}$ on M given by Theorem 3. It can not vanish on any set of positive measure, since any function supported on such a set would be in the kernel of the operator consisting of multiplication by $\left((A+iI)^{-1}\right)^{\sim}$.

Thus the function

$$\tilde{A} := \left[\left((A+iI)^{-1}\right)^{\sim}\right]^{-1} - i$$

is finite almost everywhere on M relative to the measure μ although it might (and generally will) be unbounded. Our plan is to show that under the unitary isomorphism W the operator A goes over into multiplication by $\tilde{A}$.

Proposition 4 *$x \in D(A)$ if and only if $\tilde{A}Wx \in L_2(M,\mu)$.*

Proof. Suppose $x \in D(A)$. Then $x = (A+iI)^{-1}y$ for some $y \in H$ and so

$$Wx = \left((A+iI)^{-1}\right)\tilde{}\, f, \qquad f = Wy.$$

But

$$\tilde{A}\left((A+iI)^{-1}\right)\tilde{} = 1 - ih$$

where

$$h = \left((A+iI)^{-1}\right)\tilde{}$$

is a bounded function. Thus $\tilde{A}Wx \in L_2(M,\mu)$.

Conversely, if $\tilde{A}Wx \in L_2(M, \mu)$, then $(\tilde{A}+iI)Wx \in L_2(M, \mu)$, which means that there is a $y \in H$ such that $Wy = (\tilde{A} + iI)Wx$. Therefore

$$\left((A + iI)^{-1}\right)\tilde{}(\tilde{A} + iI)Wx = Wx$$

and hence

$$x = (A + iI)^{-1}y \in D(A).$$

QED

Proposition 5 *If $h \in W(D(A))$ then $\tilde{A}h = WAW^{-1}h$.*

Proof. Let $x = W^{-1}h$ which we know belongs to $D(A)$ so we may write $x = (A + iI)^{-1}y$ for some $y \in H$, and hence

$$Ax = y - ix \quad \text{and } Wy = \left[\left((A + iI)^{-1}\right)\tilde{}\,\right]^{-1} h.$$

So

$$\begin{aligned} WAx &= Wy - iWx \\ &= \left[\left((A + iI)^{-1}\right)\tilde{}\,\right]^{-1} h - ih \\ &= \tilde{A}h \quad \text{QED} \end{aligned}$$

The function $\tilde{A}$ must be real valued almost everywhere since if its imaginary part were positive (or negative) on a set U of positive measure, then $(\tilde{A}\mathbf{1}_U, \mathbf{1}_U)_2$ would have non-zero imaginary part contradicting the fact that multiplication by $\tilde{A}$ is a self adjoint operator, being unitarily equivalent to the self adjoint operator A.

Putting all this together we get

Theorem 4 *Let A be a self adjoint operator on a separable Hilbert space H. Then there exists a finite measure space (M, μ), a unitary isomorphism $W : H \to L_2(M, \mu)$ and a real valued measurable function $\tilde{A}$ which is finite almost everywhere such that $x \in D(A)$ if and only if $\tilde{A}Wx \in L_2(M, \mu)$ and if $h \in W(D(A))$ then $\tilde{A}h = WAW^{-1}h$.*

The functional calculus.

Let f be any bounded Borel function defined on $\mathbf{R}$. Then

$$f \circ \tilde{A}$$

is a bounded function defined on M. Multiplication by this function is a bounded operator on $L_2(M, \mu)$ and hence corresponds to a bounded self-adjoint operator on H. With a slight abuse of language we might denote this operator by $O(f \circ \tilde{A})$. However we will use the more suggestive notation

$$f(A).$$

The map

$$f \mapsto f(A)$$

- is an algebraic homomorphism,
- $\overline{f}(A) = f(A)^*$,
- $\|f(A)\| \leq \|f\|_\infty$ where the norm on the left is the uniform operator norm and the norm on the right is the sup norm on $\mathbf{R}$
- if $Ax = \lambda x$ then $f(A)x = f(\lambda)x$,
- if $f \geq 0$ then $f(A) \geq 0$ in the operator sense,
- if $f_n \to f$ pointwise and if $\|f_n\|_\infty$ is bounded, then $f_n(A) \to f(A)$ strongly, and
- if f_n is a sequence of Borel functions on the line such that $|f_n(\lambda)| \leq |\lambda|$ for all n and for all $\lambda \in \mathbf{R}$, and if $f_n(\lambda) \to \lambda$ for each fixed $\lambda \in \mathbf{R}$ then for each $x \in D(A)$

$$f_n(A)x \to Ax.$$

All of the above statements are obvious except perhaps for the last two which follow from the dominated convergence theorem. It is also clear from the preceding discussion that the map $f \mapsto f(A)$ is uniquely determined by the above properties.

Multiplication by the function $e^{it\tilde{A}}$ is a unitary operator on $L_2(M, \mu)$ and

$$e^{is\tilde{A}}e^{it\tilde{A}} = e^{i(s+t)\tilde{A}}.$$

Hence from the above we conclude

Theorem 5 [Half of Stone's theorem.] *For an self adjoint operator A the operator e^{itA} given by the functional calculus as above is a unitary operator and*

$$t \mapsto e^{itA}$$

is a one parameter group of unitary transformations.

The full Stone's theorem asserts that any unitary one parameter groups is of this form. We will discuss this later.

Resolutions of the identity

For each measurable subset X of the real line we can consider its indicator function $\mathbf{1}_X$ and hence $\mathbf{1}_X(A)$ which we shall denote by $P(X)$. In other words

$$P(X) := \mathbf{1}_X(A).$$

It follows from the above that

$$\begin{aligned}
P(X)^* &= P(X) \\
P(X)P(Y) &= P(X \cap Y) \\
P(X \cup Y) &= P(X) + P(Y) \text{ if } X \cap Y = 0 \\
P(X) &= s - \lim \sum P(X_i) \text{ if } X_i \cap X_j = \emptyset \text{ if } i \neq j \text{ and } X = \bigcup X_i
\end{aligned}$$

$$\begin{aligned}
P(\emptyset) &= 0 \\
P(\mathbf{R}) &= I.
\end{aligned}$$

One then writes the spectral theorem as

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda. \tag{14}$$

In the next lecture we shall give an alternative proof of this formula which does not depend on either the Gelfand representation theorem or any of the limit theorems of Lebesgue integration. Instead, it depends on the Riesz-Dunford extension of the Cauchy theory of integration of holomorphic functions along curves to operator valued holomorphic functions.

For each $x, y \in H$ we have the complex valued measure

$$P_{x,y}(X) = (P(X)x, y)$$

and for any bounded Borel function f we have

$$(f(A)x, y) = \int_{\mathbf{R}} f(\lambda) dP_{x,y}.$$

If g is an unbounded (complex valued) Borel function on $\mathbf{R}$ we define $D(g(A))$ to consist of those $x \in H$ for which

$$\int_{\mathbf{R}} |g(\lambda)|^2 dP_{x,x} < \infty.$$

The set of such x is dense in H and we define $g(A)$ on $D(A)$ by

$$(g(A)x, y) = \int_{\mathbf{R}} g(\lambda) dP_{x,y}$$

for $x, y \in D(g(A))$ (and the Riesz representation theorem). This is written symbolically as

$$g(A) = \int_{\mathbf{R}} g(\lambda) dP.$$

In the special case $g(\lambda) = \lambda$ we write

$$A = \int_{\mathbf{R}} \lambda dP.$$

this is the spectral theorem for self adjoint operators.

In the older literature one often sees the notation

$$E_\lambda := P(-\infty, \lambda).$$

A translation of the properties of P into properties of E is

$$\begin{aligned}
E_\lambda^2 &= E_\lambda &(8)\\
E_\lambda^* &= E_\lambda &(9)\\
\lambda < \mu &\Rightarrow E_\lambda E_\mu = E_\lambda &(10)\\
\lambda_{n\to} - \infty &\Rightarrow E_{\lambda_n} \to 0 \text{ strongly} &(11)\\
\lambda_n \to +\infty &\Rightarrow E_{\lambda_n} \to I \text{ strongly} &(12)\\
\lambda_n \nearrow \lambda &\Rightarrow E_n \to E_\lambda \text{ strongly.} &(13)
\end{aligned}$$

One then writes the spectral theorem as

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda. \tag{14}$$

Stone's formula.

$$\text{s-}\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_a^b [R(\lambda - i\epsilon, T) - R(\lambda + i\epsilon, T)]\, d\lambda =$$

$$\frac{1}{2}\left(P((a,b)) + P([a,b])\right).$$

We proved this formula by a "real variables proof" via the Gelfand transform. Our second proof of the spectral theorem, due to Lorch, will be via a complex variables interpretation and proof of Stone's formula.

The Riesz-Dunford calculus.

Suppose that we have a continuous map $z \mapsto S_z$ defined on some open set of complex numbers, where S_z is a bounded operator on some fixed Banach space and by continuity, we mean continuity relative to the uniform metric on operators. If C is a continuous piecewise differentiable (or more generally any rectifiable) curve lying in this open set, and if $t \mapsto z(t)$ is a piecewise smooth (or rectifiable) parametrization of this curve, then the map $t \mapsto S_{z(t)}$ is continuous.

For any partition $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ of the unit interval we can form the Cauchy approximating sum

$$\sum_{i=1}^{n} S_{z(t_i}(z(t_i) - z(t_{i-1}),$$

and the usual proof of the existence of the Riemann integral shows that this tends to a limit as the mesh becomes more and more refined and the mesh distance tends to zero. The limit is denoted by

$$\int_C S_z dz$$

and this notation is justifies because the change of variables formula for an ordinary integral shows that this value does not depend on the parametrization, but only on the orientation of the curve C.

We are going to apply this to $S_z = R_z$, the resolvent of an operator, and the main equations we shall use are the resolvent equation (5) and the power series for the resolvent (6) which we repeat here:

$$R_z - R_w = (w - z)R_z R_w$$

and

$$R_w = R_z(I + (z - w)R_z + (z - w)^2 R_z^2 + \cdots).$$

We proved that the resolvent of a self-adjoint operator exists for all non-real values of z.