Math 212b Lecture 2

Lorch's proof of the spectral theorem.

The Riesz-Dunford calculus.

Suppose that we have a continuous map $z \mapsto S_z$ defined on some open set of complex numbers, where S_z is a bounded operator on some fixed Banach space and by continuity, we mean continuity relative to the uniform metric on operators. If C is a continuous piecewise differentiable (or more generally any rectifiable) curve lying in this open set, and if $t \mapsto z(t)$ is a piecewise smooth (or rectifiable) parametrization of this curve, then the map $t \mapsto S_{z(t)}$ is continuous.

For any partition $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ of the unit interval we can form the Cauchy approximating sum

$$\sum_{i=1}^{n} S_{z(t_i}(z(t_i) - z(t_{i-1}),$$

and the usual proof of the existence of the Riemann integral shows that this tends to a limit as the mesh becomes more and more refined and the mesh distance tends to zero. The limit is denoted by

$$\int_C S_z dz$$

and this notation is justifies because the change of variables formula for an ordinary integral shows that this value does not depend on the parametrization, but only on the orientation of the curve C.

We are going to apply this to $S_z = R_z$, the resolvent of an operator, and the main equations we shall use are the resolvent equation (5) and the power series for the resolvent (6) which we repeat here:

$$R_z - R_w = (w - z)R_z R_w$$

and

$$R_w = R_z(I + (z - w)R_z + (z - w)^2 R_z^2 + \cdots).$$

We proved that the resolvent of a self-adjoint operator exists for all non-real values of z.

But a lot of the theory goes over for the resolvent

$$R_z = R(z,T) = (zI - T)^{-1}$$

where T is an arbitrary operator on a Banach space, so long as we restrict ourselves to the resolvent set, i.e. the set where the resolvent exists as a bounded operator. So, following Lorch *Spectral Theory* we first develop some facts about integrating the resolvent in the more general Banach space setting (where our principal application will be to the case where T is a bounded operator).

For example, suppose that C is a simple closed curve contained in the disk of convergence about z of (6) i.e. of the above power series for R_w. Then we can integrate the series term by term. But

$$\int_C (z-w)^n dw = 0$$

for all $n \neq -1$ so

$$\int_C R_w dw = 0.$$

By the usual method of breaking any any deformation up into a succession of small deformations and then breaking any small deformation up into a sequence of small "rectangles" we conclude

Theorem 6 *If two curves C_0 and C_1 lie in the resolvent set and are homotopic by a family C_t of curves lying entirely in the resolvent set then*

$$\int_{C_0} R_z dz = \int_{C_1} R_z dz.$$

Here are some immediate consequences of this elementary result.

Suppose that T is a bounded operator and $|z| > \|T\|$. Then

$$(zI - T)^{-1} = z^{-1}(I - z^{-1}T)^{-1} = z^{-1}(I + z^{-1}T + z^{-2}T^2 + \cdots)$$

exists because the series in parentheses converges in the uniform metric. In other words, all points in the complex plane outside the disk of radius $\|T\|$ lie in the resolvent set of T. From this it follows that the spectrum of any bounded operator can not be empty (if the Banach space is not $\{0\}$). (Recall the the spectrum is the complement of the resolvent set.) Indeed, if the resolvent set were the whole plane, then the circle of radius zero about the origin would be homotopic to a circle of radius $> \|T\|$ via a homotopy lying entirely in the resolvent set. Integrating R_z around the circle of radius zero gives 0. We can integrate around a large circle using the above power series. In performing this integration, all terms vanish except the first which give $2\pi i I$ by the usual Cauchy integral (or by direct computation). Thus $2\pi I = 0$ which is impossible in a non-zero vector space.

Here is another very important (and easy) consequence of the preceding theorem:

Theorem 7 *Let C be a simple closed rectifiable curve lying entirely in the resolvent set of T. Then*

$$P := \frac{1}{2\pi i}\int_C R_z dz \tag{15}$$

is a projection which commutes with T, i.e.

$$P^2 = P \quad \text{and} \quad PT = TP.$$

$$R_z - R_w = (w - z)R_z R_w. \tag{5}$$

Proof. Choose a simple closed curve C' disjoint from C but sufficiently close to C so as to be homotopic to C via a homotopy lying in the resolvent set. Thus

$$P = \frac{1}{2\pi i}\int_{C'} R_w dw$$

and so

$$(2\pi i)^2 P^2 = \int_C R_z dz \int_{C'} R_w dw = \int_C \int_{C'} (R_w - R_z)(z-w)^{-1} dw dz$$

where we have used the resolvent equation (5). We write this last expression as a sum of two terms,

$$\int_{C'} R_w \int_C \frac{1}{z-w} dz dw - \int_C R_z \int_{C'} \frac{1}{z-w} dw dz.$$

$(2\pi i)^2 P^2 =$

$$\int_{C'} R_w \int_C \frac{1}{z-w} dz dw - \int_C R_z \int_{C'} \frac{1}{z-w} dw dz.$$

Suppose that we choose C' to lie entirely inside C. Then the first expression above is just $(2\pi i) \int_{C'} R_w dw$ while the second expression vanishes, all by the elementary Cauchy integral of $1/(z-w)$. Thus we get

$$(2\pi i)^2 P^2 = (2\pi i)^2 P$$

or $P^2 = P$. This proves that P is a projection. It commutes with T because it is an integral whose integrand R_z commutes with T for all z. QED

The same argument proves

Theorem 8 *Let C and C' be simple closed curves each lying in the resolvent set, and let P and P' be the corresponding projections given by (15). Then $PP' = 0$ if the curves lie exterior to one another while $PP' = P'$ if C' is interior to C.*

Let us write

$$B' := PB, \quad B'' = (I - P)B$$

for the images of the projections P and $I - P$ where P is given by (15). Each of these spaces is invariant under T and hence under R_z because $PT = TP$ and hence $PR_z = R_zP$.

For any transformation S commuting with P let us write

$$S' := PS = SP = PSP \quad \text{and}$$

$$S'' = (I-P)S = S(I-P) = (I-P)S(I-P)$$

so that S' and S' are the restrictions of S to B' and B'' respectively.

For example, we may consider $R'_z = PR_z = R_zP$. For $x' \in B'$ we have $R'_z(zI-T')x' = R_zP(zI-TP)x' = R_z(zI-T)Px' = x'$.

In other words R'_z is the resolvent of T' (on B') and similarly for R''_z. So if z is in the resolvent set for T it is in the resolvent set for T' and T''.

Conversely, suppose that z is in the resolvent set for both T' and T''. Then there exists an inverse A_1 for $zI' - T'$ on B' and an inverse A_2 for $zI'' - T''$ on B'' and so $A_1 \oplus A_2$ is the inverse of $zI - T$ on $B = B' \oplus B''$.

So a point belongs to the resolvent set of T if and only if it belongs to the resolvent set of T' and of T''. Since the spectrum is the complement of the resolvent set, we can say that a point belongs to the spectrum of T if and only if it belongs either to the spectrum of T' or of T'':

$$\mathrm{Spec}(T) = \mathrm{Spec}(T') \cup \mathrm{Spec}(T'').$$

Decomposing the spectrum.

We have thus proved

Theorem 9 *Let T be a bounded linear transformation on a Banach space and C a simple closed curve lying in its resolvent set. Let P be the projection given by (15) and*

$$B = B' \oplus B'', \qquad T = T' \oplus T''$$

the corresponding decomposition of B and of T. Then $\mathrm{Spec}(T')$ consists of those points of $\mathrm{Spec}(T)$ which lie inside C and $\mathrm{Spec}(T'')$ consists of those points of $\mathrm{Spec}(T)$ which lie exterior to C.

We now show that this decomposition is in fact the decomposition of $\mathrm{Spec}(T)$ into those points which lie inside C and outside C.

So we must show that if z lies exterior to C then it lies in the resolvent set of T'. This will certainly be true if we can find a transformation A on B which commutes with T and such that

$$A(zI - T) = P \tag{16}$$

for then A' will be the resolvent at z of T'. Now

$$(zI - T)R_w = (wI - T)R_w + (z - w)R_w = I + (z - w)R_w$$

so

$$(zI - T) \cdot \frac{1}{2\pi i} \int_C R_w \cdot \frac{1}{z - w} dw =$$

$$= \frac{1}{2\pi i} \int_C \frac{1}{z - w} dw \cdot I + \frac{1}{2\pi i} \int_C R_w dw = 0 + P = P.$$

Relation to Stone's formula.

We now begin to have a better understanding of Stone's formula: Suppose A is a self-adjoint operator. We know that its spectrum lies on the real axis. If we draw a rectangle whose upper and lower sides are parallel to the axis, and if its vertical sides do not intersect $\mathrm{Spec}(A)$, we would get a projection onto a subspace M of our Hilbert space which is invariant under A, and such that the spectrum of A when restricted to M lies in the interval cut out on the real axis by our rectangle. The problem is how to make sense of this procedure when the vertical edges of the rectangle might cut through the spectrum, in which case the integral (15) might not even be defined. This is resolved by the method of Lorch (the exposition is taken from his book) which we explain in the next section.

We have thus proved

Theorem 9 *Let T be a bounded linear transformation on a Banach space and C a simple closed curve lying in its resolvent set. Let P be the projection given by (15) and*

$$B = B' \oplus B'', \qquad T = T' \oplus T''$$

the corresponding decomposition of B and of T. Then $\operatorname{Spec}(T')$ *consists of those points of* $\operatorname{Spec}(T)$ *which lie inside C and* $\operatorname{Spec}(T'')$ *consists of those points of* $\operatorname{Spec}(T)$ *which lie exterior to C.*

Positive operators

Recall that if A is a bounded self-adjoint operator on a Hilbert space H then we write $A \geq 0$ if $(Ax, x) \geq 0$ for all $x \in H$ and (by a slight abuse of language) call such an operator positive. Clearly the sum of two positive operators is positive as is the multiple of a positive operator by a non-negative number. Also we write $A_1 \geq A_2$ for two self adjoint operators if $A_1 - A_2$ is positive.

Proposition 6 *If A is a bounded self-adjoint operator and $A \geq I$ then A^{-1} exists and*

$$\|A^{-1}\| \leq 1.$$

Proof. We have

$$\|Ax\| \, \|x\| \geq (Ax, x) \geq (x, x) = \|x\|^2$$

so

$$\|Ax\| \geq \|x\| \quad \forall \; x \in H.$$

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So A is injective, and hence A^{-1} is defined on im A and is bounded by 1 there. We must show that this image is all of H.

If y is orthogonal to im A we have

$$(x, Ay) = (Ax, y) = 0 \;\; \forall\, x \in H$$

so $Ay = 0$ so $(y, y) \leq (Ay, y) = 0$ and hence $y = 0$. Thus im A is dense in H.

Suppose that $Ax_n \to z$. Then the x_n form a Cauchy sequence by the estimate above on $\|A^{-1}\|$ and so $x_n \to x$ and the continuity of A implies that $Ax = z$. QED

Suppose that $A \geq 0$. Then for any $\lambda > 0$ we have $A+\lambda I \geq \lambda I$, and by the proposition $(A+\lambda I)^{-1}$ exists, i.e. $-\lambda$ belongs to the resolvent set of A. So we have proved.

Proposition 7 *If* $A \geq 0$ *then* $\mathrm{Spec}(A) \subset [0, \infty)$.

Theorem 10 *If A is a self-adjoint transformation then*

$$\|A\| \leq 1 \quad \Leftrightarrow \quad -I \leq A \leq I. \tag{17}$$

Proof. Suppose $\|A\| \leq 1$. Then using Cauchy-Schwarz and then the definition of $\|A\|$ we get

$$([I-A]x, x) = (x, x) - (Ax, x) \geq$$

$$\|x\|^2 - \|Ax\| \, \|x\| \geq \|x\|^2 - \|A\| \|x\|^2 \geq 0$$

Conversely, suppose that $-I \leq A \leq I$. Since $I - A \geq 0$ we know that $\mathrm{Spec}(A) \subset (-\infty, 1]$ and since $I + A \geq 0$ we have $\mathrm{Spec}(A) \subset (-1, \infty]$. So

$$\mathrm{Spec}(A) \subset [-1, 1]$$

so that the spectral radius of A is ≤ 1. But for self adjoint operators we have $\|A^2\| = \|A\|^2$ and hence the formula for the spectral radius gives $\|A\| \leq 1$. QED

An important corollary:

An immediate corollary of the theorem is the following: Suppose that μ is a real number. Then $\|A - \mu I\| \leq \epsilon$ is equivalent to $(\mu - \epsilon)I \leq A \leq (\mu + \epsilon)I$.

So one way of interpreting the spectral theorem

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda$$

is to say that for any doubly infinite sequence

$$\cdots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

with $\lambda_{-n} \to -\infty$ and $\lambda_n \to \infty$ there is a corresponding Hilbert space direct sum decomposition

$$H = \bigoplus H_i$$

invariant under A and such that the restriction of A to H_i satisfies

$$\lambda_i I \leq A_{|H_i} \leq \lambda_{i+1} I.$$

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If $\mu_i := \frac{1}{2}(\lambda_i + \lambda_{i+1})$ then another way of writing the preceding inequality is

$$\|A_{|H_i} - \mu_i I\| \leq \frac{1}{2}(\lambda_{i+1} - \lambda_i).$$

The point spectrum

We now let A denote an arbitrary (not necessarily bounded) self adjoint transformation. We say that λ belongs to the **point spectrum** of A if there exists an $x \in D(A)$ such that $x \neq 0$ and $Ax = \lambda x$. In other words if λ is an eigenvalue of A. Notice that eigenvectors corresponding to distinct eigenvalues are orthogonal: if $Ax = \lambda x$ and $Ay = \mu y$ then

$$\lambda(x, y) = (\lambda x, y) = (Ax, y) = (x, Ay) = (x, \mu y) = \mu(x, y)$$

implying that $(x, y) = 0$ if $\lambda \neq \mu$.

Also, the fact that a self-adjoint operator is closed implies that the space of eigenvectors corresponding to a fixed eigenvalue is a closed subspace of H. We let N_λ denote the space of eigenvectors corresponding to an eigenvalue λ.

Recall the statement of the spectral theorem:

In the older literature one often sees the notation

$$E_\lambda := P(-\infty, \lambda).$$

A translation of the properties of P into properties of E is

$$E_\lambda^2 = E_\lambda \tag{8}$$

$$E_\lambda^* = E_\lambda \tag{9}$$

$$\lambda < \mu \Rightarrow E_\lambda E_\mu = E_\lambda \tag{10}$$

$$\lambda_{n\to} - \infty \Rightarrow E_{\lambda_n} \to 0 \text{ strongly} \tag{11}$$

$$\lambda_n \to +\infty \Rightarrow E_{\lambda_n} \to I \text{ strongly} \tag{12}$$

$$\lambda_n \nearrow \lambda \Rightarrow E_n \to E_\lambda \text{ strongly.} \tag{13}$$

One then writes the spectral theorem as

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda. \tag{14}$$

Operators with pure point spectrum.

We say that A has **pure point spectrum** if its eigenvectors span H, in other words if

$$H = \bigoplus N_{\lambda_i}$$

where the λ_i range over the set of eigenvalues of A. Suppose that this is the case. Then let

$$M_\lambda := \bigoplus_{\mu<\lambda} N_\mu$$

where this denotes the Hilbert space direct sum, i.e. the closure of the algebraic direct sum. Let E_λ denote projection onto M_λ. Then it is immediate that the E_λ satisfy (8)-(13) and that (14) holds with the interpretation given in the preceding section. We thus have a proof of the spectral theorem for operators with pure point spectrum.

Partition into pure types.

Now consider a general self-adjoint operator A, and let

$$H_1 := \bigoplus N_\lambda$$

(Hilbert space direct sum) and set

$$H_2 := H_1^\perp.$$

The space H_1 and hence the space H_2 are invariant under A in the sense that A maps $D(A) \cap H_1$ to H_1 and similarly for H_2.

We let P denote orthogonal projection onto H_1 so $I - P$ is orthogonal projection onto H_2. We claim that

$$P[D(A)] = D(A) \cap H_1 \quad \text{and} \quad (I-P)[D(A)] = D(A) \cap H_2. \tag{18}$$

Suppose that $x \in D(A)$. We must show that $Px \in D(A)$ for then $x = Px + (I - P)x$ is a decomposition of every element of $D(A)$ into a sum of elements of $D(A) \cap H_1$ and $D(A) \cap H_2$.

By definition, we can find an orthonormal basis of H_1 consisting of eigenvectors u_i of A, and then

$$Px = \sum a_i u_i \qquad a_i := (x, u_i).$$

The sum on the right is (in general) infinite. Let y denote any finite partial sum. Since eigenvectors belong to $D(A)$ we know that $y \in D(A)$. We have

$$(A[x - y], Ay) - ([x - y], A^2 y) = 0$$

since $x - y$ is orthogonal to all the eigenvectors occurring in the expression for y. We thus have

$$\|Ax\|^2 = \|A(x - y)\|^2 + \|Ay\|^2$$

From this we see (as we let the number of terms in y increase) that both y converges to Px and the Ay converge. Hence $Px \in D(A)$ proving (18).

Let A_1 denote the operator A restricted to $P[D(A)] = D(A) \cap H_1$ with similar notation for A_2. We claim that A_1 is self adjoint (as is A_2). Clearly $D(A_1) := P(D(A))$ is dense in H_1, for if there were a vector $y \in H_1$ orthogonal to $D(A_1)$ it would be orthogonal to $D(A)$ in H which is impossible. Similarly $D(A_2) := D(A) \cap H_2$ is dense in H_2.

Now suppose that y_1 and z_1 are elements of H_1 such that

$$(A_1 x_1, y_1) = (x_1, z_1) \ \ \forall \ x_1 \in D(A_1).$$

Since $A_1 x_1 = A x_1$ and $x_1 = x - x_2$ for some $x \in D(A)$, and since y_1 and z_1 are orthogonal to x_2, we can write the above equation as

$$(Ax, y_1) = (x, z_1) \quad \forall \ x \in D(A)$$

which implies that $y_1 \in D(A) \cap H_1 = D(A_1)$ and $A_1 y_1 = A y_1 = z_1$.

In other words, A_1 is self-adjoint. Similarly, so is A_2. We have thus proved

Theorem 11 *Let A be a self-adjoint transformation on a Hilbert space H. Then*

$$H = H_1 \oplus H_2$$

with self-adjoint transformations A_1 on H_1 having pure point spectrum and A_2 on H_2 having no point spectrum such that

$$D(A) = D(A_1) \oplus D(A_2)$$

and

$$A = A_1 \oplus A_2.$$

We have proved the spectral theorem for a self adjoint operator with pure point spectrum. Our proof of the full spectral theorem will be complete once we prove it for operators with no point spectrum.

The "physical" meaning of the decomposition.

In a later lecture we will show that for the types of self-adjoint operators that arise in atomic physics or quantum chemistry, the vectors belonging to the space H in Theorem 11 are the "bound states" - those electronic states that remain within a finite distance of the nuclei for all time, while the elements of H are the "scattering states" those electronic states corresponding to electrons which remain in a bounded region for only a finite amount of time.

Completion of the proof.

In this subsection we will assume that A is a self-adjoint operator with no point spectrum, i.e. no eigenvalues.

Let $\lambda < \mu$ be real numbers and let C be a closed piecewise smooth curve in the complex plane which is symmetrical about the real axis and cuts the real axis at non-zero angle at the two points λ and μ (only). Let $m > 0$ and $n > 0$ be positive integers, and let

$$K_{\lambda\mu}(m,n) := \frac{1}{2\pi i}\int_C (z-\lambda)^m (z-\mu)^n R_z \, dz. \tag{19}$$

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In fact, we would like to be able to consider the above integral when $m = n = 0$, in which case it should give us a projection onto a subspace where $\lambda I \leq A \leq \mu I$. But unfortunately if λ or μ belong to $\mathrm{Spec}(A)$ the above integral need not converge with $m = n = 0$. However we do know that $\|R_z\| \leq (|\mathrm{im}\ z|)^{-1}$ so that the blow up in the integrand at λ and μ is killed by $(z-\lambda)^m$ and $(\mu - z)^n$ since the curve makes non-zero angle with the real axis. Since the curve is symmetric about the real axis, the (bounded) operator $K_{\lambda\mu}(m,n)$ is self-adjoint. Furthermore, modifying the curve C to a curve C' lying inside C, again intersecting the real axis only at the points λ and μ and having these intersections at non-zero angles does not change the value: $K_{\lambda\mu}(m,n)$.

We will now prove a succession of facts about $K_{\lambda\mu}(m,n)$:

$$K_{\lambda\mu}(m,n)\cdot K_{\lambda\mu}(m',n') = K_{\lambda\mu}(m+m',n+n'). \tag{20}$$

Proof. Calculate the product using a curve C' for $K_{\lambda\mu}(m',n')$ as indicated above. Then use the functional equation for the resolvent and Cauchy's integral formula exactly as in the proof of Theorem 7: $(2\pi i)^2 K_{\lambda\mu}(m,n)\cdot K_{\lambda\mu}(m',n') =$

$$\int_C \int_{C'} (z-\lambda)^m (\mu - z)^n (w-\lambda)^{m'} (\mu - w)^{n'} \frac{1}{z-w}[R_w - R_z]dzdw$$

which we write as a sum of two integrals, the first giving $(2\pi i)^2 K_{\lambda\mu}(m+m',n+n')$ and the second giving zero. QED

A similar argument (similar to the proof of Theorem 8) shows that

$$K_{\lambda\mu}(m,n) \cdot K_{\lambda'\mu'}(m',n') = 0 \;\text{ if } (\lambda,\mu) \cap (\lambda',\mu') = \emptyset. \tag{21}$$

Proposition 8 *There exists a bounded self-adjoint operator $L_{\lambda\mu}(m,n)$ such that*

$$L_{\lambda\mu}(m,n)^2 = K_{\lambda\mu}(m,n).$$

Proof. The function $z \mapsto (z-\lambda)^{m/2}(\mu - z)^{n/2}$ is defined and holomorphic on the complex plane with the closed intervals $(-\infty,\lambda]$ and $[\mu,\infty)$ removed. The integral

$$L_{\lambda\mu}(m,n) = \frac{1}{2\pi i}\int_C (z-\lambda)^{m/2}(\mu - z)^{n/2} R_z dz$$

is well defined since, if $m = 1$ or $n = 1$ the singularity is of the form $|\mathrm{im}\, z|^{-\frac{1}{2}}$ at worst which is integrable. Then the proof of (20) applies to prove the proposition. QED

For each complex z we know that $R_z x \in D(A)$. Hence

$$(A - \lambda I)R_z x = (A - zI)R_z x + (z - \lambda)R_z x = x + (z - \lambda)R_z x.$$

By writing the integral defining $K_{\lambda\mu}(m, n)$ as a limit of approximating sums, we see that $(A - \lambda I)K_{\lambda\mu}(m, n)$ is defined and that it is given by the sum of two integrals, the first of which vanishes and the second gives $K_{\lambda\mu}(m + 1, n)$.

We have thus shown that $K_{\lambda\mu}(m, n)$ maps H into $D(A)$ and

$$(A - \lambda I)K_{\lambda\mu}(m, n) = K_{\lambda\mu}(m + 1, n). \tag{22}$$

Similarly

$$(\mu I - A)K_{\lambda\mu}(m, n) = K_{\lambda\mu}(m, n + 1). \tag{23}$$

We also have

$$\lambda(x,x) \leq (Ax,x) \leq \mu(x,x) \quad \text{for } x \in \text{ im } K_{\lambda\mu}(m,n). \tag{24}$$

Proof. We have

$$([A-\lambda I]K_{\lambda\mu}(m,n)y, K_{\lambda\mu}(m,n)y)$$

$$\begin{aligned}
&= (K_{\lambda\mu}(m+1,n)y, K_{\lambda\mu}(m,n)y) \\
&= (K_{\lambda\mu}(m,n)K_{\lambda\mu}(m+1,n)y, y) \\
&= (K_{\lambda\mu}(2m+1,2n)y, y) \\
&= (L_{\lambda\mu}(2m+1,2n)^2 y, y) \\
&= (L_{\lambda\mu}(2m+1,2n)y, L_{\lambda\mu}(2m+1,2n)y) \geq 0. \\
&\Rightarrow A \geq \lambda I \quad \text{on im } K_{\lambda\mu}(m,n).
\end{aligned}$$

A similar argument shows that $A \leq \mu I$ there. QED

Thus if we define $M_{\lambda\mu}(m,n)$ to be the closure of im $K_{\lambda\mu}(m,n)$ we see that A is bounded when restricted to $M_{\lambda\mu}(m,n)$ and

$$\lambda I \leq A \leq \mu I$$

there.

We let $N_{\lambda\mu}(m,n)$ denote the kernel of $K_{\lambda\mu}(m,n)$ so that $M_{\lambda\mu}(m,n)$ and $N_{\lambda\mu}(m,n)$ are the orthogonal complements of one another.

So far we have not made use of the assumption that A has no point spectrum. Here is where we will use this assumption: Since

$$(A-\lambda I)K_{\lambda\mu}(m,n) = K_{\lambda\mu}(m+1,n)$$

we see that if $K_{\lambda\mu}(m+1,n)x = 0$ we must have $(A-\lambda I)K_{\lambda\mu}(m,n)x = 0$ which, by our assumption implies that $K_{\lambda\mu}(m,n)x = 0$. In other words,

Proposition 9 *The space $N_{\lambda\mu}(m,n)$, and hence its orthogonal complement $M_{\lambda\mu}(m,n)$ is independent of m and n.*

We will denote the common space $M_{\lambda\mu}(m,n)$ by $M_{\lambda\mu}$. We have proved that A is a bounded operator when restricted to $M_{\lambda\mu}$ and satisfies

$$\lambda I \leq A \leq \mu I \quad \text{on } M_{\lambda\mu}$$

there.

We now claim that

Prop: If $\lambda < \nu < \mu$ then $\quad M_{\lambda\mu} = M_{\lambda\nu} \oplus M_{\nu\mu}. \qquad (25)$

Proof. Let $C_{\lambda\mu}$ denote the rectangle of height one parallel to the real axis and cutting the real axis at the points λ and μ. Use similar notation to define the rectangles $C_{\lambda\nu}$ and $C_{\nu\mu}$. Consider the integrand

$$S_z := (z-\lambda)(z-\mu)(z-\nu)R_z$$

and let

$$T_{\lambda\mu} := \frac{1}{2\pi i}\int_{C_{\lambda\mu}} S_z dz$$

with similar notation for the integrals over the other two rectangles of the same integrand. Then clearly

$$T_{\lambda\mu} = T_{\lambda\nu} + T_{\nu\mu} \quad \text{and } T_{\lambda\nu}\cdot T_{\nu\mu} = 0. \tag{26}$$

Also, writing $zI - A = (z-\nu)I + (\nu I - A)$ we see that

$$(\nu I - A)K_{\lambda\nu}(1,1) = T_{\lambda\mu}$$

Since A has no point spectrum, the closure of the image of $T_{\lambda\mu}$ is the same as the closure of the image of $K_{\lambda\mu}(1,1)$, namely $M_{\lambda\mu}$. The proposition now follows from (26).

$$\text{If } \lambda < \nu < \mu \text{ then } \quad M_{\lambda\mu} = M_{\lambda\nu} \oplus M_{\nu\mu}. \tag{25}$$

If we now have a doubly infinite sequence as in our reformulation of the spectral theorem, and we set $M_i := M_{\lambda_i \lambda_{i+1}}$ we have proved the spectral theorem (in the no point spectrum case - and hence in the general case) if we show that

$$\bigoplus M_i = H.$$

In view of (25) it is enough to prove that the closure of the limit of M_{-rr} is all of H as $r \to \infty$, or, what amounts to the same thing, if y is perpendicular to all $K_{-rr}(1,1)x$ then y must be zero. Now

$$(K_{-rr}(1,1)x, y) = (x, K_{-rr}(1,1)y)$$

so we must show that if $K_{-rr}y = 0$ for all r then $y = 0$. Now

$$K_{-rr} = \frac{1}{2\pi i}\int_C (z+r)(r-z)R_z dz = -\frac{1}{2\pi i}\int (z^2 - r^2)R_z$$

where we may take C to be the circle of radius r centered at the origin.

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where we may take C to be the circle of radius r centered at the origin. We also have

$$1 = \frac{1}{2\pi i r^2}\int_C \frac{r^2-z^2}{z}dz.$$

So

$$y = \frac{1}{2\pi i r^2}\int_C (r^2-z^2)[z^{-1}I - R_z]dz \cdot y.$$

Now $(zI-A)R_z = I$ so $-AR_z = I - zR_z$ or

$$z^{-1}I - R_z = -z^{-1}AR_z$$

so (pulling the A out from under the integral sign) we can write the above equation as

$$y = Ag_r \quad \text{where } g_r = \frac{1}{2\pi i r^2}\int_C (r^2-z^2)z^{-1}R_z dz \cdot y.$$

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Now on C we have $z = re^{i\theta}$ so $z^2 = r^2 e^{2i\theta} = r^2(\cos 2\theta + i \sin 2\theta)$ and hence

$$z^2 - r^2 = r^2(\cos 2\theta - 1 + i \sin 2\theta) = 2r^2(-\sin^2\theta + i \sin\theta\cos\theta).$$

Now $\|R_z\| \leq |r \sin\theta|^{-1}$ so we see that

$$\|(z^2 - r^2)R_z\| \leq 4r.$$

Since $|z^{-1}| = r^{-1}$ on C, we can bound $\|g_r\|$ by

$$\|g_r| \leq (2\pi r^2)^{-1} \cdot r^{-1} \cdot 4r \cdot 2\pi r\|y\| = 4r^{-1}\|y\| \to 0$$

as $r \to \infty$. Since $y = Ag_r$ and A is closed (being self-adjoint) we conclude that $y = 0$. This concludes Lorch's proof of the spectral theorem.

In about two weeks I am going to need the following technical fact about operators with purely continuous spectrum. Here is as good a place as any to prove it:

Suppose that A is a self-adjoint operator with only continuous spectrum. Let

$$E_\mu := P((-\infty, \mu))$$

be its spectral resolution. For any $\psi \in \mathcal{H}$ the function

$$\mu \mapsto (E_\mu \psi, \psi)$$

is continuous. It is also a monotone increasing function of μ. For any $\epsilon > 0$ we can find a sufficiently negative a such that $|(E_a\psi, \psi)| < \epsilon/2$ and a sufficiently large b such that $\|\psi\|^2 - (E_n\psi, \psi) < \epsilon/2$. On the compact interval $[a, b]$ any continuous function is uniformly continuous. Therefore the function $\mu \mapsto (E_\mu\psi, \psi)$ is uniformly continuous on $\mathbb{R}$.

Now let ϕ and ψ be elements of $\mathcal{H}$ and consider the product measure

$$d(E_\lambda \phi, \phi) d(E_\mu \psi, \psi)$$

on the $\mathcal{R}^2$, the λ, μ plane.

Lemma 1 *The diagonal line $\lambda = \mu$ has measure zero relative to the above product measure.*

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Proof. We may assume that $\phi \neq 0$. For any $\epsilon > 0$ we can find a $\delta > 0$ such that

$$(E_{\mu+\delta}\psi, \psi) - E_{\mu-\delta}\psi, \psi) < \frac{\epsilon}{\|\phi\|^2}$$

for all $\mu \in \mathbb{R}$. So

$$\int_{\mathbb{R}} d(E_\lambda \phi, \phi \int_{\lambda-\delta}^{\lambda+\delta} (E_\mu \psi, \psi) < \epsilon.$$

This says that the measure of the band of width δ about the diagonal has measure less that ϵ. Letting δ shrink to 0 shows that the diagonal line has measure zero. $\square$

We can restate this lemma more abstractly as follows: Consider the Hilbert space $\mathcal{H}\hat{\otimes}\mathcal{H}$ (the completion of the tensor product). The E_λ and E_μ determine a projection valued measure Q on the plane with values in $\mathcal{H}\hat{\otimes}\mathcal{H}$. The spectral measure associated with the operator $A\otimes I - I\otimes A$ is then $F_\rho := Q(\{\lambda,\mu|\lambda-\mu<\rho\})$. So an abstract way of formulating the lemma is

Proposition 1 *0 is not an eigenvalue of $A\otimes I - I\otimes A$ on $\mathcal{H}\hat{\otimes}\mathcal{H}$.* ●