Math 212b Lecture 3

The Cayley transform and self-adjointness.

First, an example of the computation of the resolvent.

An important example.

Consider the operator $\quad H_0 : \quad L_2(\mathbf{R}^3) \to L_2(\mathbf{R}^3)$

given by

$$H_0 := \Delta := -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right).$$

Here the domain of H_0 is taken to be those $\phi \in L_2(\mathbf{R}^3)$ for which the differential operator on the right, taken in the distributional sense, when applied to ϕ gives an element of $L_2(\mathbf{R}^3)$.

The operator H_0 has a fancy name. It is called the "free Hamiltonian of non-relativistic quantum mechanics". Strictly speaking we should add "for particles of mass one-half in units where Planck's constant is one".

Using the Fourier transform.

The Fourier transform is a unitary isomorphism of $L_2(\mathbf{R}^3)$ into $L_2(\mathbf{R}_3)$ and carries H_0 into multiplication by ξ^2 whose domain consists of those $\hat{\phi} \in L_2(\mathbf{R}_3)$ such that $\xi^2\hat{\phi}(\xi)$ belongs to $L_2(\mathbf{R}_3)$. The operators

$$V(t) : L_2(\mathbf{R}_3) \to L_2(\mathbf{R}_3), \qquad \hat{\phi}(\xi) \mapsto e^{-it\xi^2}\hat{\phi}$$

form a one parameter group of unitary transformations whose infinitesimal generator in the sense of Stone's theorem is operator consisting of multiplication by ξ^2 with domain as given above. [The minus sign before the i in the exponential is the convention used in quantum mechanics. So we write $\exp -itA$ for the one-parameter group associated to the self-adjoint operator A. I apologize for this (rather irrelevant) notational change, but I want to make the notation in this section consistent with what you will see in physics books.]

The resolvent at points along the negative real axis.

Thus the operator of multiplication by ξ^2, and hence the operator H_0 is a self-adjoint transformation. The operator of multiplication by ξ^2 is clearly non-negative and so every point on the negative real axis belongs to its resolvent set. Let us write a point on the negative real axis as $-\mu^2$ where $\mu > 0$. Then the resolvent of multiplication by ξ^2 at such a point on the negative real axis is given by multiplication by $-f$ where

$$f(\xi) = f_\mu(\xi) := \frac{1}{\mu^2 + \xi^2}.$$

1. The operator H_0 is self adjoint.

2. The one parameter group of unitary transformations it generates via Stone's theorem is

$$U(t) = \mathcal{F}^{-1} V(t) \mathcal{F}$$

where $V(t)$ is multiplication by $e^{-it\xi^2}$.

3. Any point $-\mu^2, \quad \mu > 0$ lies in the resolvent set of H_0 and

$$R(-\mu^2, H_0) = -\mathcal{F}^{-1} m_f \mathcal{F}$$

where m_f denotes the operation of multiplication by f and f is as given above.

4. If $g \in \mathcal{S}$ and m_g denotes multiplication by g, then the the operator $\mathcal{F}^{-1} m_g \mathcal{F}$ consists of convolution by $\breve{g}$. Neither the function $e^{-it\xi^2}$ nor the function f belongs to $\mathcal{S}$, so the operators $U(t)$ and $R(-\mu^2, H_0)$ can only be thought of as convolutions in the sense of generalized functions.

A more explicit formula for the resolvent.

Nevertheless, we will be able to give some slightly more explicit (and very instructive) representations of these operators as convolutions. For example, we will use the Cauchy residue calculus to compute $\check{f}$ and we will find, up to factors of powers of 2π that $\check{f}$ is the function

$$Y_\mu(x) := \frac{e^{-\mu r}}{r}$$

where r denotes the distance from the origin, i.e. $r^2 = x^2$. This function has an integrable singularity at the origin, and vanishes rapidly at infinity. So convolution by Y_μ will be well defined and given by the usual formula on elements of $\mathcal{S}$ and extends to an operator on $L_2(\mathbf{R}^3)$.

The Yukawa potential.

The function Y_μ is known as the **Yukawa potential**. Yukawa introduced this function in 1934 to explain the forces that hold the nucleus together. The exponential decay with distance contrasts with that of the ordinary electromagnetic or gravitational potential $1/r$ and, in Yukawa's theory, accounts for the fact that the nuclear forces are short range. In fact, Yukawa introduced a "heavy boson" to account for the nuclear forces. The role of mesons in nuclear physics was predicted by brilliant theoretical speculation well before any experimental discovery. Here are the details:

Since $\check{f} \in L_2$ we can compute its inverse Fourier transform as

$$(2\pi)^{-3/2}\check{f} = \lim_{R\to\infty}(2\pi)^{-3}\int_{|\xi|\le R}\frac{e^{i\xi\cdot x}}{\mu^2+\xi^2}d\xi. \qquad (1)$$

Here lim means the L_2 limit and $|\xi|$ denotes the length of the vector ξ, i.e. $|\xi| = \sqrt{\xi^2}$ and we will use similar notation $|x| = r$ for the length of x. Assume $x \neq 0$. Let

$$u := \frac{\xi \cdot x}{|\xi||x|}$$

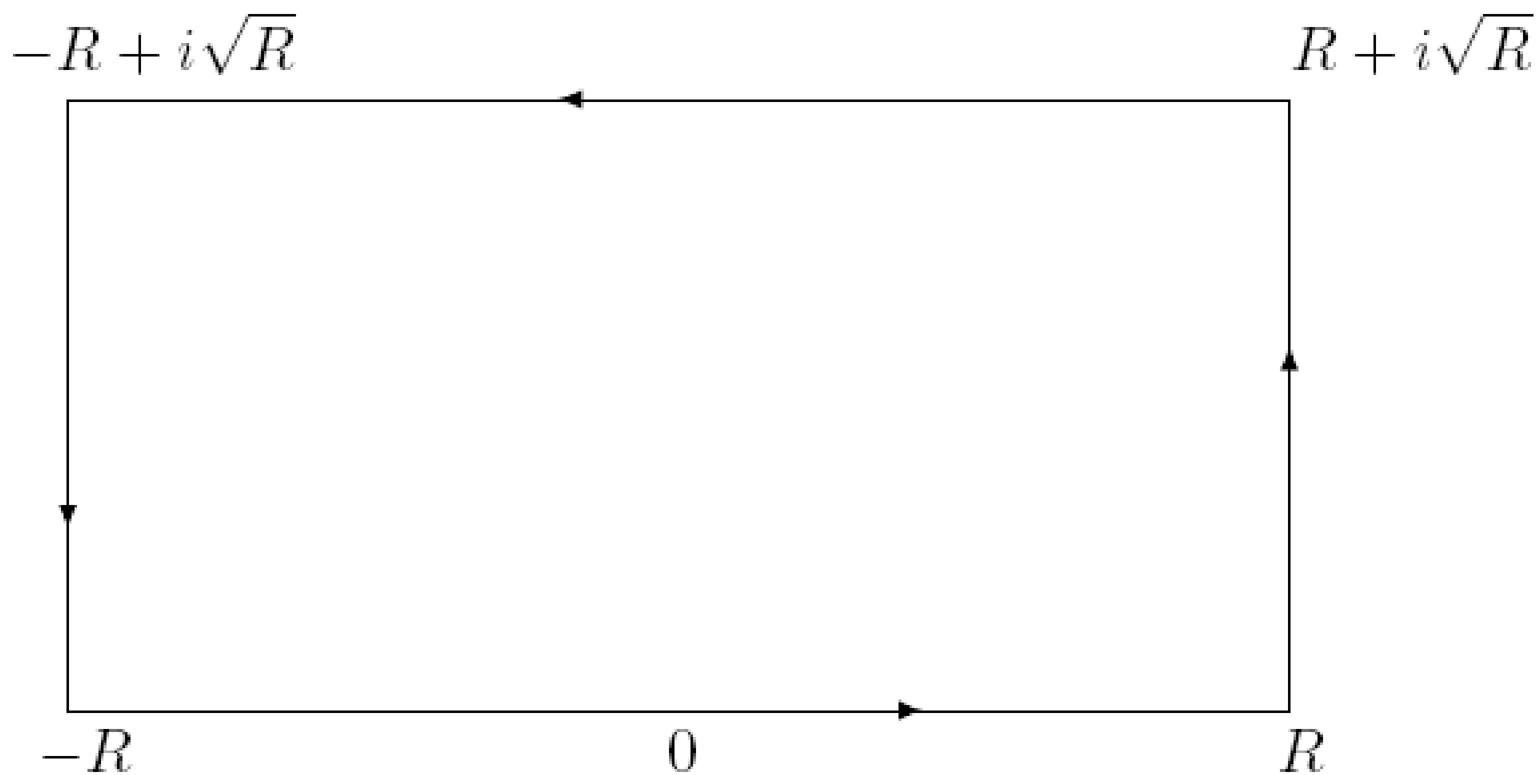
so u is the cosine of the angle between x and ξ. Fix x and introduce spherical coordinates in ξ space with x at the north pole and $s = |\xi|$ so that

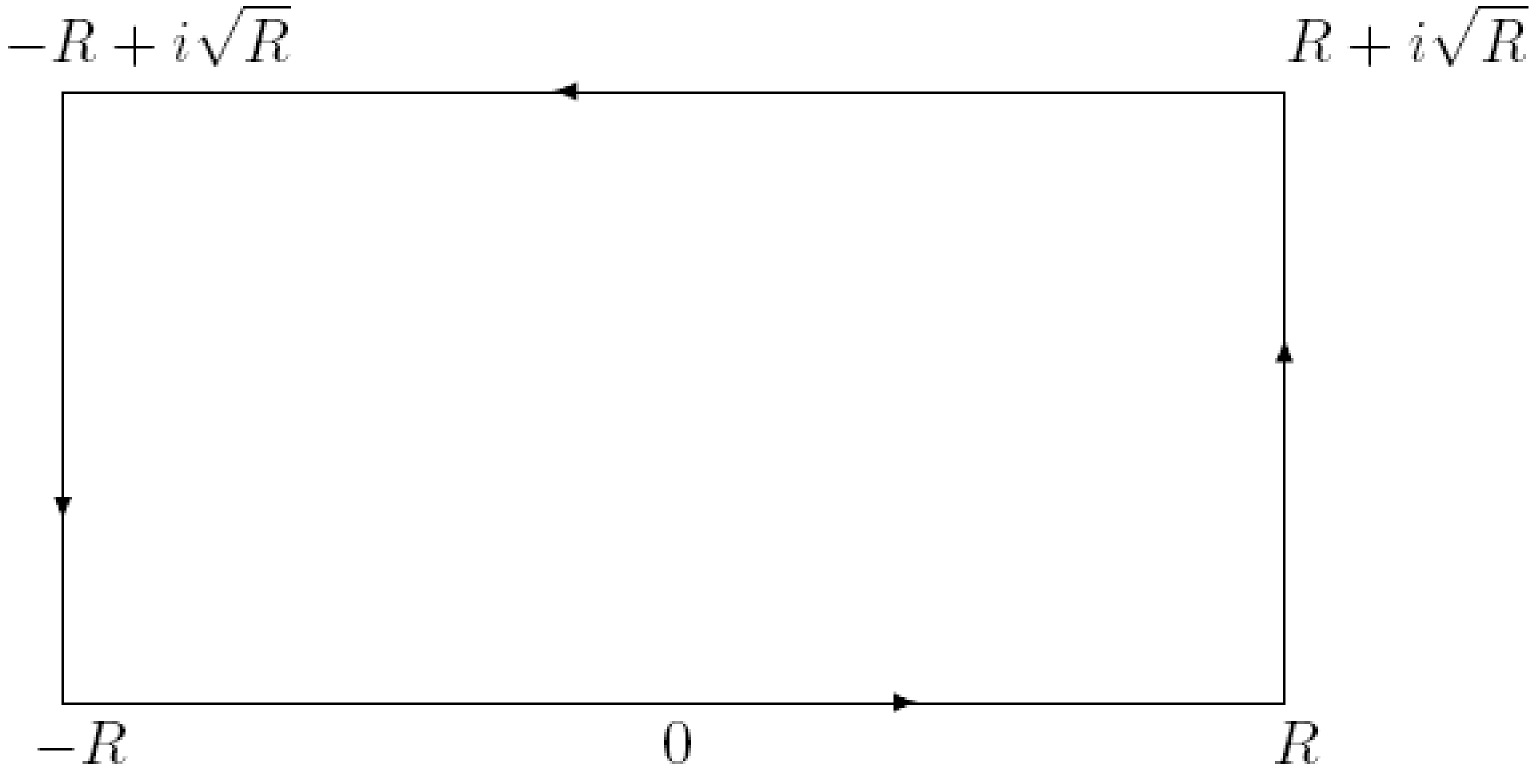
$$(2\pi)^{-3}\int_{|\xi|\leq R}\frac{e^{i\xi\cdot x}}{\mu^2+\xi^2}d\xi = (2\pi)^{-2}\int_0^R\int_{-1}^1\frac{e^{is|x|u}}{s^2+\mu^2}s^2duds$$

$$=$$

$$\frac{1}{(2\pi)^2 i|x|}\int_{-R}^R\frac{se^{is|x|}}{(s+i\mu)(s-i\mu)}ds.$$

This last integral is along the bottom of the path in the complex s-plane consisting of the boundary of the rectangle as drawn in the figure.





On the two vertical sides of the rectangle, the integrand is bounded by some constant time $1/R$, so the contribution of the vertical sides is $O(1/\sqrt{R})$. On the top the integrand is $O(e^{-\sqrt{R}})$. So the limits of these integrals are zero. There is only one pole in the upper half plane at $s = i\mu$, so the integral is given by $2\pi i\times$ this residue which equals

$$2\pi i \frac{i\mu e^{-\mu|x|}}{2i\mu} = \pi i e^{-\mu|x|}.$$

$$(2\pi)^{-3/2}\check{f} = \lim_{R\to\infty}(2\pi)^{-3}\int_{|\xi|\le R}\frac{e^{i\xi\cdot x}}{\mu^2+\xi^2}d\xi. \qquad (1)$$

$$2\pi i\frac{i\mu e^{-\mu|x|}}{2i\mu} = \pi i e^{-\mu|x|}.$$

Inserting this back into (1) we see that the limit exists and is equal to

$$(2\pi)^{-3/2}\hat{f} = \frac{1}{4\pi}\frac{e^{-\mu|x|}}{|x|}.$$

We conclude that for $\phi \in \mathcal{S}$

$$[(H_0+\mu^2)^{-1}\phi](x) = \frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{e^{-\mu|x-y|}}{|x-y|}\phi(y)dy,$$

and since $(H_0+\mu^2)^{-1}$ is a bounded operator on L_2 this formula extends in the L_2 sense to L_2.

The time evolution of the free Hamiltonian.

The "explicit" calculation of the operator $U(t)$ is slightly more tricky. The function $\xi \mapsto e^{-it\xi^2}$ is an "imaginary Gaussian", so we expect is inverse Fourier transform to also be an imaginary Gaussian, and then we would have to make sense of convolution by a function which has absolute value one at all points. There are several ways to proceed. One involves integration by parts, and I hope to explain how this works later on in the course in conjunction with the method of stationary phase.

Here I will follow Reed-Simon vol II p.59 and add a little positive term to t and then pass to the limit. In other words, let α be a complex number with positive real part and consider the function

$$\xi \mapsto e^{-\xi^2\alpha}$$

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This function belongs to $\mathcal{S}$ and its inverse Fourier transform is given by the function

$$x \mapsto (2\alpha)^{-3/2} e^{-x^2/4\alpha}.$$

(In fact, we verified this when α is real, but the integral defining the inverse Fourier transform converges in the entire half plane Re $\alpha > 0$ uniformly in any Re $\alpha > \epsilon$ and so is holomorphic in the right half plane. So the formula for real positive α implies the formula for α in the half plane.)

We thus have

$$(e^{-H_0 \alpha} \phi)(x) = \left(\frac{1}{4\pi\alpha} \right)^{3/2} \int_{\mathbf{R}^3} e^{-|x-y|^2/4\alpha} \phi(y) dy.$$

Here the square root in the coefficient in front of the integral is obtained by continuation from the positive square root on the positive axis. For example, if we take $\alpha = \epsilon + it$ so that $-\alpha = -i(t - i\epsilon)$ we get

$$(U(t)\phi)(x) = \lim_{\epsilon \searrow 0} (U(t - i\epsilon)\phi)(x)$$

$$= \lim_{\epsilon \searrow 0} (4\pi i(t - i\epsilon))^{-\frac{3}{2}} \int e^{-|x-y|^2/4i(t-i\epsilon)} \phi(y) dy.$$

Here the limit is in the sense of L_2. We thus could write

$$(U(t))(\phi)(x) = (4\pi i)^{-3/2} \int e^{i|x-y|^2/4t} \phi(y) dy$$

if we understand the right hand side to mean the $\epsilon \searrow 0$ limit of the preceding expression.

Recall that if A is a self-adjoint operator on a Hilbert space $\mathbf{H}$ we can form the one parameter group of unitary operators

$$U(t) = e^{iAt}$$

by virtue of a functional calculus which allows us to construct $f(A)$ for any bounded Borel function defined on $\mathbf{R}$ (if we use our first proof of the spectral theorem using the Gelfand representation theorem) or for any function holomorphic on $\operatorname{Spec}(A)$ if we use our second proof. In any event, the spectral theorem allows us to write

$$U(t) = \int_{-\infty}^{\infty} e^{it\lambda} dE_\lambda$$

and to verify that

$$U(0) = I, \quad U(s+t) = U(s)U(t)$$

and that U depends continuously on t. We called this assertion the first half of Stone's theorem.

The second half (to be stated more precisely below) asserts the converse: that any one parameter group of unitary transformations can be written in either, hence both, of the above forms.

The idea that we will follow hinges on the following elementary computation

$$\int_0^\infty e^{(-z+ix)t}dt = \left.\frac{e^{(-z+ix)t}}{-z+ix}\right|_{t=0}^{\infty} = \frac{1}{z-ix} \text{ if Re } z > 0$$

valid for any real number x. If we substitute A for x and write $U(t)$ instead of e^{iAt} this suggests that

$$R(z, iA) = (zI - iA)^{-1} = \int_0^\infty e^{-zt}U(t)dt \text{ if Re } z > 0.$$

Since A is self-adjoint, its spectrum is real. So the spectrum of iA is purely imaginary, and hence any z not on the imaginary axis is in the resolvent set of iA. The above formula gives us an expression for the resolvent in terms of $U(t)$ for z lying in the right half plane. We can obtain a similar formula for the left half plane.

Our previous studies encourage us to believe that once we have found all these putative resolvents, it should not be so hard to reconstruct A and then the one-parameter group $U(t) = e^{iAt}$.

This program works! But because of some of the subtleties involved in the definition of a self-adjoint operator, we will begin with an important theorem of von-Neumann which we will need, and which will also greatly clarify exactly what it means to be self-adjoint.

A second matter which will lengthen these proceedings is that while we are at it, we will prove a more general version of Stone's theorem valid in an arbitrary Frechet space **F** and for "uniformly bounded semigroups" rather than unitary groups. Stone proved his theorem to meet the needs of quantum mechanics, where a unitary one parameter group corresponds, via *Wigner's theorem* to a one parameter group of symmetries of the logic of quantum mechanics. In more pedestrian terms, unitary one parameter groups arise from solutions of Schrodinger's equation. But many other important equations, for example the heat equations in various settings, require the more general result.

The treatment here will essentially follow that of Yosida, *Functional Analysis* especially Chapter IX, Nelson, *Topics in dynamics I: Flows*, and Reed and Simon *Methods of Mathematical Physics, II. Fourier Analysis, Self-Adjointness*.

von Neumann's Cayley transform.

The group $Gl(2, \mathbf{C})$ of all invertible complex two by two matrices acts as "fractional linear transformations" on the plane: the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ sends } \quad z \mapsto \frac{az+b}{cz+d}.$$

Two different matrices M_1 and M_2 give the same fractional linear transformation if and only if $M_1 = \lambda M_2$ for some (non-zero complex) number λ as is clear from the definition. Since

$$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} = 2i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the fractional linear transformations corresponding to $\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$ are inverse to one another.

It is a theorem in the elementary theory of complex variables that fractional linear transformations are the only orientation preserving transformations of the plane which carry circles and lines into circles and lines. Even without this general theory, an immediate computation shows that $\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ carries the (extended) real axis onto the unit circle, and hence its inverse carries the unit circle onto the extended real axis. ("Extended" means with the point ∞ added.) Indeed in the expression

$$z = \frac{x - i}{x + i}$$

when x is real, the numerator is the complex conjugate of the denominator and hence $|z| = 1$. Under this transformation, the cardinal points $0, 1, \infty$ of the extended real axis are mapped as follows:

$$0 \mapsto -1, \quad 1 \mapsto -i, \quad \text{and} \quad \infty \mapsto 1.$$

We might think of (multiplication by) a real number as a self-adjoint transformation on a one dimensional Hilbert space, and (multiplication by) a number of absolute value one as a unitary operator on a one dimensional Hilbert space. This suggests in general that if A is a self adjoint operator, then

$$(A - iI)(A + iI)^{-1}$$

should be unitary. In fact, we can be much more precise. First some definitions:

An operator U, possibly defined only on a subspace of a Hilbert space $\mathbf{H}$ is called **isometric** if

$$\|Ux\| = \|x\|$$

for all x in its domain of definition.

Symmetric operators.

Recall that in order to define the adjoint T^* of an operator T it is necessary that its domain $D(T)$ be dense in $\mathbf{H}$. Otherwise the equation

$$(Tx, y) = (x, T^*y) \;\forall\; x \in D(T)$$

does not determine T^*y. A transformation T (in a Hilbert space $\mathbf{H}$) is called **symmetric** if $D(T)$ is dense in $\mathbf{H}$ so that T^* is defined and

$$D(T) \subset D(T^*) \quad \text{and } Tx = T^*x \quad \forall\; x \in D(T).$$

Another way of saying the same thing is T is symmetric if $D(T)$ is dense and

$$(Tx, y) = (x, Ty) \quad \forall\; x, y \in D(T).$$

Symmetric versus self-adjoint.

A self-adjoint transformation is symmetric since $D(T) = D(T^*)$ is one of the requirements of being self-adjoint. Exactly how and why a symmetric operator can fail to be self-adjoint will be clarified in the ensuing discussion. All of the results of this section are due to von Neumann.

Theorem 1 *Let T be a closed symmetric operator. Then $(T + iI)x = 0$ implies that $x = 0$ for any $x \in D(T)$ so $(T + iI)^{-1}$ exists as an operator on its domain*

$$D\left[(T + iI)^{-1}\right] = \mathrm{im}(T + iI).$$

This operator is bounded on its domain and the operator

$$U_T := (T - iI)(T + iI)^{-1} \quad \text{with } D(U_T) = D\left[(T + iI)^{-1}\right] = \mathrm{im}(T + iI)$$

is isometric and closed. The operator $(I - U_T)^{-1}$ exists and

$$T = i(U_T + I)(U_T - I)^{-1}.$$

In particular, $D(T) = \mathrm{im}(I - U_T)$ is dense in H.

Conversely, if U is a closed isometric operator such that $\mathrm{im}(I - U)$ is dense in $\mathbf{H}$ then $T = i(U + I)(I - U)^{-1}$ is a symmetric operator with $U = U_T$.

Proof. For any $x \in D(T)$ we have

$$([T \pm iI]x, [T \pm iI]x) = (Tx, Tx) \pm (Tx, ix) \pm (ix, Tx) + (x, x).$$

The middle terms cancel because T is symmetric. Hence

$$\|[T \pm iI]x\|^2 = \|Tx\|^2 + \|x\|^2. \tag{1}$$

Taking the plus sign shows that $(T + iI)x = 0 \;\Rightarrow\; x = 0$ and also shows that $\|[T + iI]x\| \geq \|x\|$ so

$$\|[T + iI]^{-1}y\| \leq \|y\| \quad \text{for} \quad y \in [T + iI](D(T)).$$

If we write $x = [T + iI]^{-1}y$ then (1) shows that

$$\|U_T y\|^2 = \|Tx\|^2 + \|x\|^2 = \|y\|^2$$

so U_T is an isometry with domain consisting of all $y = (T + iI)x$, i.e. with domain $D([T + iI]^{-1}) = \mathrm{im}[T + iI]$.

$$\|[T \pm iI]x\|^2 = \|Tx\|^2 + \|x\|^2. \tag{1}$$

We now show that U_T is closed. So we must show that if $y_n \to y$ and $z_n \to z$ where $z_n = U_T y_n$ then $y \in D(U_T)$ and $U_T y = z$. The y_n form a Cauchy sequence and $y_n = [T + iI]x_n$ since $y_n \in \mathrm{im}(T + iI)$. From (1) we see that the x_n and the Tx_n form a Cauchy sequence, so $x_n \to x$ and $Tx_n \to w$ which implies that $x \in D(T)$ and $Tx = w$ since T is assumed to be closed. But then $(T+iI)x = w+ix = y$ so $y \in D(U_T)$ and $w-ix = z = U_T y$. So we have shown that U_T is closed.

Subtract and add the equations

$$\begin{aligned} y &= (T + iI)x \\ U_T y &= (T - iI)x \quad \text{to get} \\ \frac{1}{2}(I - U_T)y &= ix \quad \text{and} \\ \frac{1}{2}(I + U_T)y &= Tx. \end{aligned}$$

The third equation shows that

$$(I - U_T)y = 0 \;\Rightarrow\; x = 0 \;\Rightarrow Tx = 0 \;\Rightarrow (I + U_T)y = 0$$

by the fourth equation. So

$$y = \frac{1}{2}([I - U_T]y + [I + U_T]y) = 0.$$

Subtract and add the equations

$$\begin{aligned} y &= (T + iI)x \\ U_T y &= (T - iI)x \quad \text{to get} \\ \frac{1}{2}(I - U_T)y &= ix \quad \text{and} \\ \frac{1}{2}(I + U_T)y &= Tx. \end{aligned}$$

Thus $(I - U_T)^{-1}$ exists, and $y = (I - U_T)^{-1}(2ix)$ from the third of the four equations above, and the last equation gives

$$Tx = \frac{1}{2}(I + U_T)y = \frac{1}{2}(I + U_T)(I - U_T)^{-1}2ix$$

or

$$T = i(I + U_T)(I - U_T)^{-1}$$

as required. Furthermore, every $x \in D(T)$ is in $\mathrm{im}(I - U_T)$. This completes the proof of the first half of the theorem.

Now suppose we start with an isometry U and suppose that $(I-U)y = 0$ for some $y \in D(U)$. Let $z \in \text{im}(I-U)$ so $z = w-Uw$ for some w. We have

$$(y,z) = (y,w)-(y,Uw) = (Uy,Uw)-(y,Uw) = (Uy-y,Uw) = 0.$$

Since we are assuming that $\text{im}(I-U)$ is dense in $\mathbf{H}$, the condition $(y,z) = 0 \ \forall\ z \in \text{im}(I-U)$ implies that $y = 0$. Thus $(I-U)^{-1}$ exists, and we may define

$$T = i(I+U)(I-U)^{-1}$$

with

$$D(T) = D\left((I-U)^{-1}\right) = \text{im}(I-U)$$

dense in $\mathbf{H}$.

$$T = i(I+U)(I-U)^{-1}$$

with

$$D(T) = D\left((I-U)^{-1}\right) = \text{im}(I-U)$$

dense in $\mathbf{H}$. Suppose that $x = (I-U)u,\ y = (I-U)v\ \in D(T) = \text{im}(I-U)$. Then

$$(Tx, y) = (i(I+U)u, (I-U)v) = i\left[(Uu, v) - (u, Uv)\right] + i\left[(u, v) - (Uu, Uv)\right]$$

The second expression in brackets vanishes since U is an isometry. So $(Tx, y) =$

$$i(Uu, v) - i(u, Uv) = (-Uu, iv) + (u, iUv) = ([I-U]u, i[I+U]v) = (x, Ty).$$

This shows that T is symmetric.

To see that $U_T = \breve{U}$ we again write $x = (I - U)u$. We have

$$Tx = i(I + U)u \text{ so } (T + iI)x = 2iu \text{ and } (T - iI)x = 2iUu.$$

Thus $D(U_T) = \{2iu \;\; u \in D(U)\} = D(U)$ and

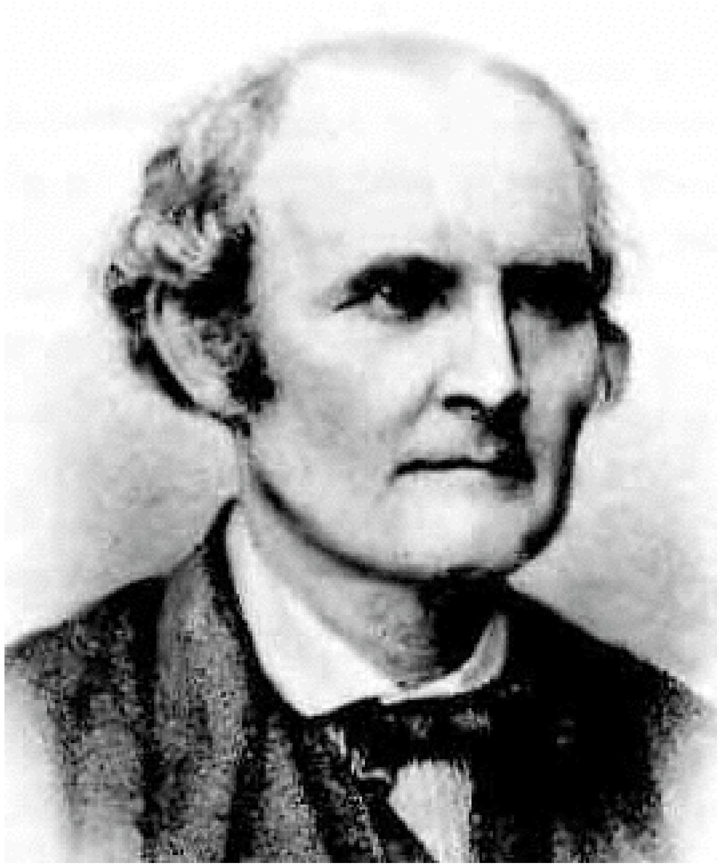
$$U_T(2iu) = 2iUu = U(2iu).$$

Thus $U = U_T$.

We must still show that T is a closed operator. T maps $x_n = (I - U)u_n$ to $(I + U)u_n$. If both $(I - U)u_n$ and $(I + U)u_n$ converge, then u_n and Uu_n converge. The fact that U is closed implies that if $u = \lim u_n$ then $u \in D(U)$ and $Uu = \lim Uu_n$. But this that $(I - U)u_n \to (I - U)u$ and $i(I + U)u_n \to i(I + U)u$ so T is closed. QED

The map $T \mapsto U_T$ from symmetric operators to isometries is called the **Cayley transform**.

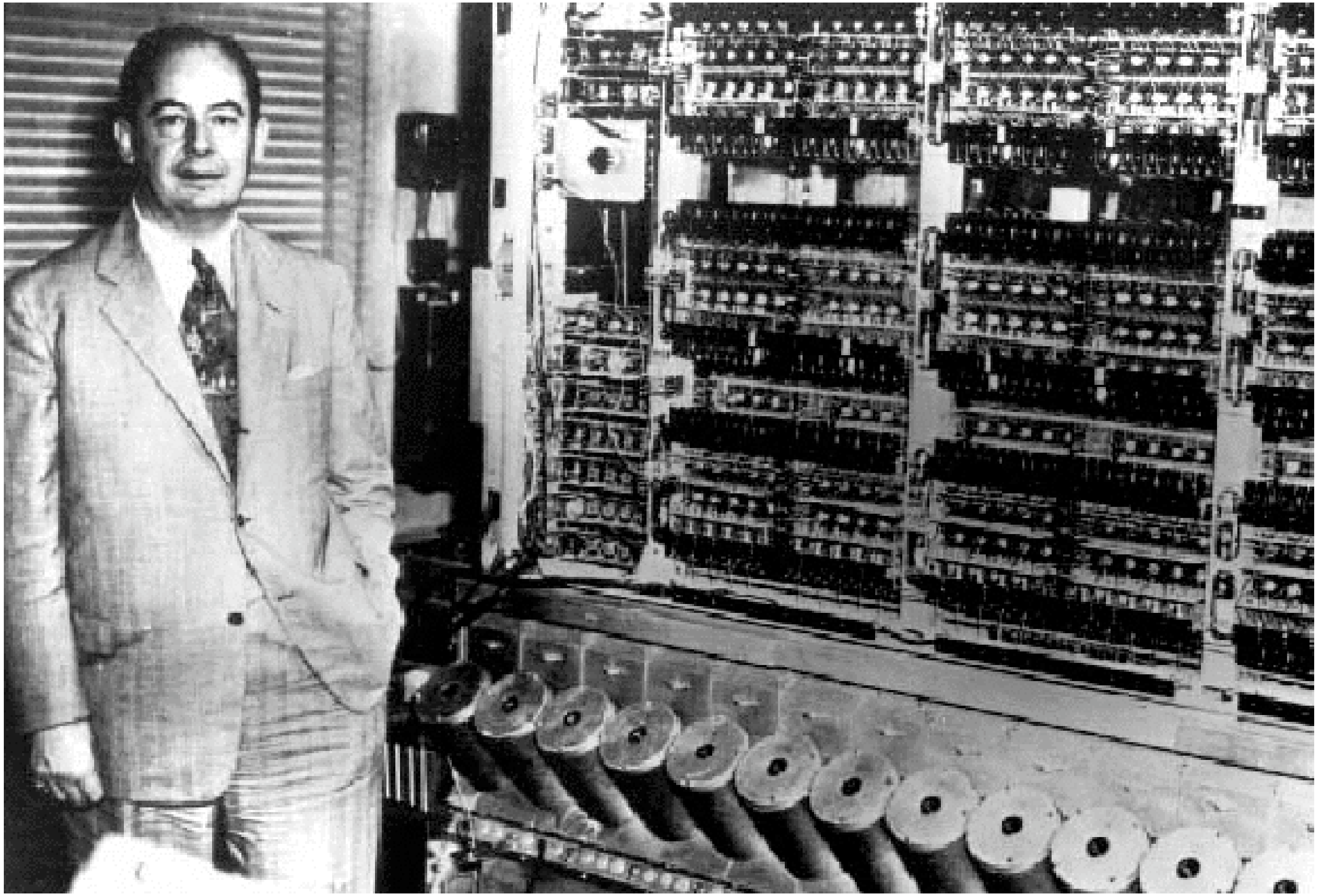
Arthur Cayley



Born: **16 Aug 1821 in Richmond, Surrey, England**
Died: **26 Jan 1895 in Cambridge, Cambridgeshire, England**

John von Neumann

Born: **28 Dec 1903 in Budapest, Hungary**
Died: **8 Feb 1957 in Washington D.C., USA**



Von Neumann with the first Institute computer

Recall that an isometry is unitary if its domain and image are all of $\mathbf{H}$. If U is a closed isometry, then $x_n \in D(U)$ and $x_n \to x$ implies that Ux_n is convergent, hence $x \in D(U)$ and $Ux = \lim Ux_n$. Similarly, if $Ux_n \to y$ then the x_n are Cauchy, hence convergent to an x with $Ux = y$. So for any closed isometry U the spaces $D(U)^\perp$ and $\mathrm{im}(U)^\perp$ measure how far U is from being unitary: If they both reduce to the zero subspace then U is unitary.

For a closed symmetric operator T define

$$\mathbf{H}_T^+ = \{x \in \mathbf{H} | T^* x = ix\} \text{ and } \mathbf{H}_T^- = \{x \in \mathbf{H} | T^* x = -ix\}. \quad (2)$$

$$\mathbf{H}_T^+ = \{x \in \mathbf{H} | T^* x = ix\} \text{ and } \mathbf{H}_T^- = \{x \in \mathbf{H} | T^* x = -ix\}. \quad (2)$$

The main theorem of this section is

Theorem 2 *Let T be a closed symmetric operator and $U = U_T$ its Cayley transform. Then*

$$\mathbf{H}_T^+ = D(U)^\perp \quad \textit{and} \quad \mathbf{H}_T^- = (\mathrm{im}(U))^\perp.$$

Every $x \in D(T^*)$ is uniquely expressible as

$$x = x_0 + x_+ + x_-$$

with $x_0 \in D(T)$, $x_+ \in \mathbf{H}_T^+$ and $x_- \in \mathbf{H}_T^-$, so

$$T^* x = T x_0 + i x_+ - i x_-.$$

In particular, T is self adjoint if and only if U is unitary.

Proof. To say that $x \in D(U)^{\perp} = D\left((T+iI)^{-1}\right)^{\perp}$ says that

$$(x,(T+iI)y) = 0 \quad \forall\ y \in D(T).$$

This says that

$$(x,Ty) = -(x,iy) = (ix,y) \quad \forall\ y \in D(T).$$

This is precisely the assertion that $x \in D(T^*)$ and $T^*x = ix$. We can read these equations backwards to conclude that $\mathbf{H}_T^+ = D(U)^{\perp}$. Similarly, if $x \in \mathrm{im}(U)^{\perp}$ then $(x,(T-iI)z) = 0 \ \ \forall z \in D(T)$ implying $T^*x = -ix$ and conversely.

We know that $D(U)$ and $\mathrm{im}(U)$ are closed subspaces of $\mathbf{H}$ so any $w \in \mathbf{H}$ can be written as the sum of an element of $D(U)$ and an element of $D(U)^{\perp}$. Taking $w = (T^*+iI)x$ for some $x \in D(T^*)$ gives

$$(T^*+iI)x = y_0 + x_1, \quad y_0 \in D(U) = \mathrm{im}(T+iI),\ x_1 \in D(U)^{\perp}.$$

$$(T^* + iI)x = y_0 + x_1, \quad y_0 \in D(U) = \mathrm{im}(T + iI),\ x_1 \in D(U)^\perp.$$

We can write $y_0 = (T + iI)x_0,\ x_0 \in D(T)$ so

$$(T^* + iI)x = (T + iI)x_0 + x_1.$$

Since $T^* = T$ on $D(T)$ and $T^*x_1 = ix_1$as $x_1 \in D(U)^\perp$ we have

$$T^*x_1 + ix_1 = 2ix_1.$$

So if we set

$$x_+ = \frac{1}{2i}x_1$$

we have

$$x_1 = (T^* + iI)x_+, \quad x_+ \in D(U)^\perp.$$

$$(T^* + iI)x = y_0 + x_1, \quad y_0 \in D(U) = \mathrm{im}(T + iI),\ x_1 \in D(U)^\perp.$$

$$x_+ = \frac{1}{2i} x_1$$

we have

$$x_1 = (T^* + iI)x_+, \qquad x_+ \in D(U)^\perp.$$

so

$$(T^* + iI)x = (T^* + iI)(x_0 + x_+)$$

or

$$T^*(x - x_0 - x_+) = -i(x - x_0 - x_+).$$

This implies that $(x - x_0 - x_+) \in \mathbf{H}_T^- = \mathrm{im}(U)^\perp$. So if we set

$$x_- := x - x_0 - x_+$$

we get the desired decomposition $x = x_0 + x_+ + x_-$.

To show that the decomposition is unique, suppose that

$$x_0 + x_+ + x_- = 0.$$

Applying $(T^* + iI)$ gives

$$0 = (T + iI)x_0 + 2ix_+.$$

But $(T + iI)x_0 \in D(U)$ and $x_+ \in D(U)^\perp$ so both terms above must be zero, so $x_+ = 0$. Also, from the preceding theorem we know that $(T + iI)x_0 = 0 \;\Rightarrow\; x_0 = 0$. Hence since $x_0 = 0$ and $x_+ = 0$ we must also have $x_- = 0$. QED

Equibounded continuous semigroups on a Frechet space.

A Frechet space $\mathbf{F}$ is a vector space with a topology defined by a sequence of semi-norms and which is complete. An important example is the Schwartz space $\mathcal{S}$. Let $\mathbf{F}$ be such a space. We want to consider a one parameter family of operators T_t on $\mathbf{F}$ defined for all $t \geq 0$ and which satisfy the following conditions:

- $T_0 = I$
- $T_t \circ T_s = T_{t+s}$
- $\lim_{t \to t_0} T_t x = T_{t_0} x \ \forall t_0 \geq 0$ and $x \in \mathbf{F}$.
- For any defining seminorm p there is a defining seminorm q and a constant K such that $p(T_t x) \leq K q(x)$ for all $t \geq 0$ and all $x \in \mathbf{F}$.

We call such a family an **equibounded continuous semigroup**. We will usually drop the adjective "continuous" and even "equibounded" since we will not be considering any other kind of semigroup.

The infinitesimal generator.

We are going to begin by showing that every such semigroup has an " infinitesimal generator", i.e. can be written in some sense as $T_t = e^{At}$. It is important to observe that we have made a serious change of convention in that we are dropping the i that we have used until now. With this new notation, for example, the infinitesimal generator of a group of unitary transformations will be a skew-adjoint operator rather than a self-adjoint operator. In quantum mechanics, where an "observable" is a self-adjoint operator, there is a good reason for emphasizing the self-adjoint operators, and hence including the i. There are many good reasons for deviating from the physicists' notation, not the least having to do with the theory of Lie algebras. I do not want to go into these reasons now. Some will emerge from the ensuing notation. But the presence or absence of the i is a cultural divide between physicists and mathematicians.

Enter the resolvent.

So we define the operator A as

$$Ax = \lim_{t \searrow 0} \frac{1}{t}(T_t - I)x.$$

That is A is the operator defined on the domain $D(A)$ consisting of those x for which the limit exists.

Our first task is to show that $D(A)$ is dense in $\mathbf{F}$. For this we begin as promised with the putative resolvent

$$R(z) := \int_0^\infty e^{-zt} T_t dt \tag{3}$$

which is defined (by the boundedness and continuity properties of T_t) for all z with $\operatorname{Re} z > 0$.

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which is defined (by the boundedness and continuity properties of T_t) for all z with Re $z > 0$. We begin by checking that every element of $\operatorname{im} R(z)$ belongs to $D(A)$: We have

$$\frac{1}{h}(T_h - I)R(z)x = \frac{1}{h}\int_0^\infty e^{-zt} T_{t+h} x dt - \frac{1}{h}\int_0^\infty e^{-zt} T_t x dt =$$

$$\frac{1}{h}\int_h^\infty e^{-z(r-h)} T_r x dr - \frac{1}{h}\int_0^\infty e^{-zt} T_t x dt$$

$$= \frac{e^{zh}-1}{h}\int_h^\infty e^{-zt} T_t x dt - \frac{1}{h}\int_0^h e^{-zt} T_t x dt$$

$$= \frac{e^{zh}-1}{h}\left[R(z)x - \int_0^h e^{-zt} T_t dt\right] - \frac{1}{h}\int_0^h e^{-zt} T_t x dt.$$

$$\frac{1}{h}(T_h - I)R(z)x = \frac{e^{zh}-1}{h}\left[R(z)x - \int_0^h e^{-zt}T_t dt\right] - \frac{1}{h}\int_0^h e^{-zt}T_t x dt.$$

If we now let $h \to 0$, the integral inside the bracket tends to zero, and the expression on the right tends to x since $T_0 = I$. We thus see that

$$R(z)x \in D(A)$$

and

$$AR(z) = zR(z) - I,$$

or, rewriting this in a more familiar form,

$$(zI - A)R(z) = I. \tag{4}$$

This equation says that $R(z)$ is a right inverse for $zI - A$. It will require a lot more work to show that it is also a left inverse.