Math 212b Lecture 4

Stone's theorem.

$$\mathbf{H}_T^+ = \{x \in \mathbf{H} | T^*x = ix\} \text{ and } \mathbf{H}_T^- = \{x \in \mathbf{H} | T^*x = -ix\}. \quad (2)$$

The main theorem of this section is

Theorem 2 *Let T be a closed symmetric operator and $U = U_T$ its Cayley transform. Then*

$$\mathbf{H}_T^+ = D(U)^\perp \quad \textit{and} \quad \mathbf{H}_T^- = (\mathrm{im}(U))^\perp.$$

Every $x \in D(T^*)$ is uniquely expressible as

$$x = x_0 + x_+ + x_-$$

with $x_0 \in D(T)$, $x_+ \in \mathbf{H}_T^+$ and $x_- \in \mathbf{H}_T^-$, so

$$T^*x = Tx_0 + ix_+ - ix_-.$$

In particular, T is self adjoint if and only if U is unitary.

Equibounded continuous semigroups on a Frechet space.

A Frechet space $\mathbf{F}$ is a vector space with a topology defined by a sequence of semi-norms and which is complete. An important example is the Schwartz space $\mathcal{S}$. Let $\mathbf{F}$ be such a space. We want to consider a one parameter family of operators T_t on $\mathbf{F}$ defined for all $t \geq 0$ and which satisfy the following conditions:

- $T_0 = I$
- $T_t \circ T_s = T_{t+s}$
- $\lim_{t \to t_0} T_t x = T_{t_0} x \; \forall t_0 \geq 0$ and $x \in \mathbf{F}$.
- For any defining seminorm p there is a defining seminorm q and a constant K such that $p(T_t x) \leq K q(x)$ for all $t \geq 0$ and all $x \in \mathbf{F}$.

We call such a family an **equibounded continuous semigroup**. We will usually drop the adjective "continuous" and even "equibounded" since we will not be considering any other kind of semigroup.

The infinitesimal generator.

We are going to begin by showing that every such semigroup has an " infinitesimal generator", i.e. can be written in some sense as $T_t = e^{At}$. It is important to observe that we have made a serious change of convention in that we are dropping the i that we have used until now. With this new notation, for example, the infinitesimal generator of a group of unitary transformations will be a skew-adjoint operator rather than a self-adjoint operator. In quantum mechanics, where an "observable" is a self-adjoint operator, there is a good reason for emphasizing the self-adjoint operators, and hence including the i. There are many good reasons for deviating from the physicists' notation, not the least having to do with the theory of Lie algebras. I do not want to go into these reasons now. Some will emerge from the ensuing notation. But the presence or absence of the i is a cultural divide between physicists and mathematicians.

Enter the resolvent.

So we define the operator A as

$$Ax = \lim_{t \searrow 0} \frac{1}{t}(T_t - I)x.$$

That is A is the operator defined on the domain $D(A)$ consisting of those x for which the limit exists.

Our first task is to show that $D(A)$ is dense in $\mathbf{F}$. For this we begin as promised with the putative resolvent

$$R(z) := \int_0^\infty e^{-zt} T_t dt \tag{3}$$

which is defined (by the boundedness and continuity properties of T_t) for all z with $\operatorname{Re} z > 0$.

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which is defined (by the boundedness and continuity properties of T_t) for all z with Re $z > 0$. We begin by checking that every element of $\operatorname{im} R(z)$ belongs to $D(A)$: We have

$$\frac{1}{h}(T_h - I)R(z)x = \frac{1}{h}\int_0^\infty e^{-zt} T_{t+h} x dt - \frac{1}{h}\int_0^\infty e^{-zt} T_t x dt =$$

$$\frac{1}{h}\int_h^\infty e^{-z(r-h)} T_r x dr - \frac{1}{h}\int_0^\infty e^{-zt} T_t x dt$$

$$= \frac{e^{zh}-1}{h}\int_h^\infty e^{-zt} T_t x dt - \frac{1}{h}\int_0^h e^{-zt} T_t x dt$$

$$= \frac{e^{zh}-1}{h}\left[R(z)x - \int_0^h e^{-zt} T_t dt\right] - \frac{1}{h}\int_0^h e^{-zt} T_t x dt.$$

$$\frac{1}{h}(T_h - I)R(z)x = \frac{e^{zh}-1}{h}\left[R(z)x - \int_0^h e^{-zt}T_t dt\right] - \frac{1}{h}\int_0^h e^{-zt}T_t x dt.$$

If we now let $h \to 0$, the integral inside the bracket tends to zero, and the expression on the right tends to x since $T_0 = I$. We thus see that

$$R(z)x \in D(A)$$

and

$$AR(z) = zR(z) - I,$$

or, rewriting this in a more familiar form,

$$(zI - A)R(z) = I. \tag{4}$$

This equation says that $R(z)$ is a right inverse for $zI - A$. It will require a lot more work to show that it is also a left inverse.

We will first prove that $D(A)$ is dense in $\mathbf{F}$ by showing that $\mathrm{im}(R(z))$ is dense. In fact, taking s to be real, we will show that

$$\lim_{s \to \infty} sR(s)x = x \quad \forall\, x \in \mathbf{F}. \tag{5}$$

Indeed,

$$\int_0^\infty se^{-st}dt = 1$$

for any $s > 0$. So we can write

$$sR(s)x - x = s\int_0^\infty e^{-st}[T_t x - x]dt.$$

Applying any seminorm p we obtain

$$p(sR(s)x - x) \leq s\int_0^\infty e^{-st}p(T_t x - x)dt.$$

$$p(sR(s)x - x) \leq s\int_0^\infty e^{-st}p(T_tx - x)dt.$$

For any $\epsilon > 0$ we can, by the continuity of T_t, find a $\delta > 0$ such that

$$p(T_tx - x) < \epsilon \ \ \forall \ \ 0 \leq t \leq \delta.$$

Now let us write

$$s\int_0^\infty e^{-st}p(T_tx - x)dt = s\int_0^\delta e^{-st}p(T_tx - x)dt + s\int_\delta^\infty e^{-st}p(T_tx - x)dt.$$

The first integral is bounded by

$$\epsilon s\int_0^\delta e^{-st}dt \leq \epsilon s\int_0^\infty e^{-st}dt = \epsilon.$$

$$s\int_0^\infty e^{-st}p(T_tx-x)dt = s\int_0^\delta e^{-st}p(T_tx-x)dt+s\int_\delta^\infty e^{-st}p(T_tx-x)dt.$$

The first integral is bounded by

$$\epsilon s\int_0^\delta e^{-st}dt \le \epsilon s\int_0^\infty e^{-st}dt = \epsilon.$$

As to the second integral, let M be a bound for $p(T_tx) + p(x)$ which exists by the uniform boundedness of T_t. The triangle inequality says that $p(T_tx - x) \le p(T_tx) + p(x)$ so the second integral is bounded by

$$M\int_\delta^\infty se^{-st}dt = Me^{-s\delta}.$$

This tends to 0 as $s \to \infty$, completing the proof that $sR(s)x \to x$ and hence that $D(A)$ is dense in $\mathbf{F}$.

The differential equation

Theorem 3 *If* $x \in D(A)$ *then for any* $t > 0$

$$\lim_{h \to 0} \frac{1}{h}[T_{t+h} - T_t]x = AT_t x = T_t Ax.$$

In colloquial terms, we can formulate the theorem as saying that

$$\frac{d}{dt} T_t = AT_t = T_t A$$

in the sense that the appropriate limits exist when applied to $x \in D(A)$.

Proof. Since T_t is continuous in t, we have

$$T_t Ax = T_t \lim_{h \searrow 0} \frac{1}{h}[T_h - I]x = \lim_{h \searrow 0} \frac{1}{h}[T_t T_h - T_t]x =$$

$$\lim_{h \searrow 0} \frac{1}{h}[T_{t+h} - T_t]x = \lim_{h \searrow 0} \frac{1}{h}[T_h - I]T_t x$$

for $x \in D(A)$. This shows that $T_t x \in D(A)$ and

$$\lim_{h \searrow 0} \frac{1}{h}[T_{t+h} - T_t]x = AT_t x = T_t Ax.$$

To prove the theorem we must show that we can replace $h \searrow 0$ by $h \to 0$. Our strategy is to show that with the information that we already have about the existence of right handed derivatives, we can conclude that

$$T_t x - x = \int_0^t T_s Ax ds.$$

Since T_t is continuous, this is enough to give the desired result.

In order to establish the above equality, it is enough, by the Hahn-Banach theorem to prove that for any $\ell \in \mathbf{F}^*$ we have

$$\ell(T_t x) - \ell(x) = \int_0^t \ell(T_s Ax) ds.$$

In turn, it is enough to prove this equality for the real and imaginary parts of ℓ.

So it all boils down to a lemma in the theory of functions of a real variable:

Lemma 1 *Suppose that f is a continuous real valued function of t with the property that the right hand derivative*

$$\frac{d^+}{dt} f := \lim_{h \searrow 0} \frac{f(t+h) - f(t)}{h} = g(t)$$

exists for all t and $g(t)$ is continuous. Then f is differentiable with $f' = g$.

Proof. We first prove that $\frac{d^+}{dt}f \geq 0$ on an interval $[a, b]$ implies that $f(b) \geq f(a)$. Suppose not. Then there exists an $\epsilon > 0$ such that

$$f(b) - f(a) < -\epsilon(b - a).$$

Set

$$F(t) := f(t) - f(a) + \epsilon(t - a).$$

Then $F(a) = 0$ and

$$\frac{d^+}{dt}F > 0.$$

At a this implies that there is some $c > a$ near a with $F(c) > 0$. On the other hand, since $F(b) < 0$, and F is continuous, there will be some point $s < b$ with $F(s) = 0$ and $F(t) < 0$ for $s < t \leq b$. This contradicts the fact that $[\frac{d^+}{dt}F](s) > 0$.

Thus if $\frac{d^+}{dt} f \geq m$ on an interval $[t_1, t_2]$ we may apply the above result to $f(t) - mt$ to conclude that

$$f(t_2) - f(t_1) \geq m(t_2 - t_1),$$

and if $\frac{d^+}{dt} f(t) \leq M$ we can apply the above result to $Mt - f(t)$ to conclude that $f(t_2) - f(t_1) \leq M(t_2 - t_1)$. So if $m = \min g(t) = \min \frac{d^+}{dt} f$ on the interval $[t_1, t_2]$ and M is the maximum, we have

$$m \leq \frac{f(t_2) - f(t_1)}{t_2 - t_1} \leq M.$$

Since we are assuming that g is continuous, this is enough to prove that f is indeed differentiable with derivative g. QED

The resolvent.

We have already verified that

$$R(z) = \int_0^\infty e^{-zt} T_t dt$$

maps $\mathbf{F}$ into $D(A)$ and satisfies

$$(zI - A)R(z) = I$$

for all z with Re $z > 0$, cf (4).

We shall now show that for this range of z

$$(zI - A)x = 0 \quad \Rightarrow \quad x = 0 \quad \forall\, x \in D(A)$$

so that $(zI - A)^{-1}$ exists and that it is given by $R(z)$.

Suppose that $Ax = zx \quad x \in D(A)$

and choose $\ell \in \mathbf{F}^*$ with $\ell(x) = 1$. Consider

$$\phi(t) := \ell(T_t x).$$

By the result of the preceding section we know that ϕ is a differentiable function of t and satisfies the differential equation

$$\phi'(t) = \ell(T_t Ax) = \ell(T_t zx) = z\ell(T_t x) = z\phi(t), \quad \phi(0) = 1.$$

So

$$\phi(t) = e^{zt}$$

which is impossible since $\phi(t)$ is a bounded function of t and the right hand side of the above equation is not bounded for $t \geq 0$ since the real part of z is positive.

We know that

$$(zI - A)R(z) = I. \tag{4}$$

We have from (4) that

$$(zI - A)R(z)(zI - A)x = (zI - A)x$$

and we know that $R(z)(zI - A)x \in D(A)$. From the injectivity of $zI - A$ we conclude that $R(z)(zI - A)x = x$.

We have already established the following:

The resolvent $R(z) = R(z, A) := \int_0^\infty e^{-zt} T_t dt$ is defined as a strong limit for Re $z > 0$ and, for this range of z:

$$\begin{aligned} D(A) &= \operatorname{im}(R(z,A)) && (6) \\ AR(z,A)x = R(z,A)Ax &= (zR(z,A) - I)x \quad x \in D(A) && (7) \\ AR(z,A)x &= (zR(z,A) - I)x \quad \forall\, x \in \mathbf{F} && (8) \\ \lim_{z \nearrow \infty} zR(z,A)x &= x \quad \text{for } z \text{ real } \ \forall x \in \mathbf{F}. && (9) \end{aligned}$$

We also have

Theorem 4 *The operator A is closed.*

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Proof. Suppose that $x_n \in D(A)$, $x_n \to x$ and $y_n \to y$ where $y_n = Ax_n$. We must show that $x \in D(A)$ and $Ax = y$. Set

$$z_n := (I - A)x_n \quad \text{so} \quad z_n \to x - y.$$

Since $R(1, A) = (I - A)^{-1}$ is a bounded operator, we conclude that

$$x = \lim x_n = \lim(I - A)^{-1} z_n = (I - A)^{-1}(x - y).$$

From (6) we see that $x \in D(A)$ and from the preceding equation that $(I - A)x = x - y$ so $Ax = y$. QED

Application to Stone's theorem.

We now have enough information to complete the proof of Stone's theorem:

Suppose that $U(t)$ is a one-parameter group of unitary transformations on a Hilbert space. We have $(U(t)x, y) = (x, U(t)^{-1}y) = (x, U(-t)y)$ and so differentiating at the origin shows that the infinitesimal generator A, which we know to be closed, is skew-symmetric:

$$(Ax, y) = (x, Ay) \quad \forall\ x, y \in D(A).$$

Also the resolvents $(zI - A)^{-1}$ exist for all z which are not purely imaginary, and $(zI - A)$ maps $D(A)$ onto the whole Hilbert space $\mathbf{H}$.

Writing $A = iT$ we see that T is symmetric and that its Cayley transform U_T has zero kernel and is surjective, i.e. is unitary. Hence T is self-adjoint. This proves Stone's theorem that every one parameter group of unitary transformations is of the form e^{iTt} with T self-adjoint.

A useful formula.

$$AR(z,A)x \quad = \quad (zR(z,A)-I)x \quad \forall\, x \in \mathbf{F} \quad (8)$$

For $r > 0$ let

$$J_r := (I - r^{-1}A)^{-1} = rR(r,A)$$

so by (8) we have

$$AJ_r = r(J_r - I). \tag{10}$$

Example: translations on the real line.

Consider the one parameter group of translations acting on $L_2(\mathbf{R})$:

$$[U(t)x](s) = x(s-t). \tag{11}$$

This is defined for all $x \in \mathcal{S}$ and is an isometric isomorphism there, so extends to a unitary one parameter group acting on $L_2(\mathbf{R})$. Equally well, we can take the above equation in the sense of distributions, where it makes sense for all elements of $\mathcal{S}'$, in particular for all elements of $L_2(\mathbf{R})$. We know that we can differentiate in the distributional sense to obtain

$$A = -\frac{d}{ds}$$

as the "infinitesimal generator" in the distributional sense.

Let us see what the general theory gives. Let $y_r := J_r x$ so

$$y_r(s) = r\int_0^\infty e^{-rt}x(s-t)dt = r\int_{-\infty}^s e^{-r(s-u)}x(u)du.$$

The right hand expression is a differentiable function of s and

$$y_r'(s) = rx(s) - r^2\int_{-\infty}^s e^{-r(s-u)}x(u)du = rx(s) - ry_r(s).$$

On the other hand we know from (10) that

$$Ay_r = AJ_r x = r(y_r - x).$$

Putting the two equations together gives

$$A = -\frac{d}{ds}$$

as expected. This is a skew-adjoint operator in accordance with Stone's theorem.

$$[U(t)x](s) = x(s-t). \tag{11}$$

We can now go back and give an intuitive explanation of what goes wrong when considering this same operator A but on $L_2[0,1]$ instead of on $L_2(\mathbf{R})$. If x is a smooth function of compact support lying in $(0,1)$, then x can not tell whether it is to be thought of as lying in $L_2([0,1])$ or $L_2(\mathbf{R})$, so the only choice for a unitary one parameter group acting on x (at least for small $t>0$) if the shift to the right as given by (11). But once t is large enough that the support of $U(t)x$ hits the right end point, 1, this transformation can not continue as is. The only hope is to have what "goes out" the right hand side come in, in some form, on the left, and unitarity now requires that

$$\int_0^1 |x(s-t)|^2 dt = \int_0^1 |x(t)|^2 dt$$

where now the shift in (11) means mod 1.

unitarity now requires that

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where now the shift in (11) means mod 1. This still allows freedom in the choice of phase between the exiting value of the x and its incoming value. Thus we specify a unitary one parameter group when we fix a choice of phase as the effect of "passing go". This choice of phase is the origin of the θ that are needed to introduce in finding the self adjoint extensions of $\frac{1}{i}\frac{d}{dt}$ acting on functions vanishing at the boundary.

Review of the Cayley transform.

Recall that an operators S with domain $\mathcal{L}$ is called **symmetric** if

$$(Sf, g) = (f, Sg) \quad \forall \quad f, g \in \mathcal{L}.$$

Proposition 1 *Let S be a symmetric operator with domain $\mathcal{L}$. Then S has a closed extension and this closed extension is also symmetric.*

Proof. Let $\mathcal{D}$ consist of all $f \in \mathcal{H}$ for which there exists a sequence $f_n \to f$ with $Sf_n \to g$. This is clearly a linear subspace which contains $\mathcal{L}$. If $h \in \mathcal{L}$ we have

$$(g, h) = \lim(Sf_n, h) = \mathrm{Range}(f_n, Sh) = (f, Sh).$$

$$(g,h) = \lim(Sf_n, h) = \text{Range}(f_n, Sh) = (f, Sh).$$

This determines g uniquely since $\mathcal{L}$ is dense. If we then define $\overline{S}$ by $\overline{S}f = g$ then the graph of $\overline{S}$ is the closure of the graph of S and so $\overline{S}$ is closed. If $h_n \in \mathcal{L}$ satisfy $h_n \to h$ and $Sh_n \to k$ then since $((\overline{S}f, h_n) = (f, Sh_n)$ we can pass to the limit to conclude that $(\overline{S}f, k) = (f, \overline{S}k)$ so $\overline{S}$ is symmetric.

The Cayley transform.

Let S be a symmetric operator with domain $\mathcal{L}$. For $f \in \mathcal{L}$ we have

$$\|(S + iI)f\|^2 = \|Sf\|^2 + \|f\|^2 = \|(S - iI)f\|^2$$

so there is an isometric operator $U := (S - iI)(S + iI)^{-1}$ called the **Cayley transform** mapping $\mathrm{im}(S + iI)$ bijectively onto $\mathrm{im}(S - iI)$.

Proposition 2 *There is a one to one correspondence between symmetric extensions of S and isometric extensions of U.*

Proof. If T is a symmetric extension of S then clearly its Cayley transform V is an isometric extension of U, the Cayley transform of S.

Suppose that V is an isometric extension of U. This means that V maps a subspace $\mathcal{M}^+ \supset \operatorname{im}(S + iI)$ isometrically onto a subspace $\mathcal{M}^- \supset \operatorname{im}(S - iI)$ which is an extension of U. We first show that $V - I$ is injective:

Suppose $f \in \mathcal{M}^+$ satisfies $Vf = f$. Let $h \in \mathcal{L}$ and set $g = (S + i)h$. Then

$$\begin{aligned} 2i(f,h) &= (f, (S - Ii)h - (S + iI)h) \\ &= (f, Ug - g) \\ &= (Vf, Vg) - (f, g) = 0. \end{aligned}$$

Since $\mathcal{L}$ is dense this shows that $f = 0$ so V is injective.

Set

$$\mathcal{D} := (V - I)\mathcal{M}^+$$

and define T on $\mathcal{D}$ by

$$Tf := \frac{1}{2}(V + I)g \quad \text{if } f = \frac{1}{2}i(V - I)g, \qquad g \in \mathcal{M}^+.$$

If $f_1 = \frac{1}{2}i(V - I)g_1$ and $f_2 = \frac{1}{2}i(V - I)g_2$ then

$$\begin{aligned}
(Tf_1, f_2) &= (\frac{1}{2}(V + I)g_1, \frac{1}{2}i(V - I)g_2) \\
&= -\frac{1}{4}i\{-(Vg_1, g_2) + (g_1, Vg_2)\} \\
&= \left(\frac{1}{2}i(V - I)g_1, \frac{1}{2}(V + I)g_2\right) \\
&= (f_1, Tf_2).
\end{aligned}$$

This shows that T is an extension of S and the definition of T can be written as

$$(T + iI)^{-1} = \frac{1}{2} i(V - I)$$

or

$$V = (T - iI)(T + iI)^{-1}$$

which says that V is the Cayley transform of T. $\square$

The deficiency indices.

For a symmetric operator S define the spaces

$$\begin{aligned}\mathcal{L}^{\pm} &= \{f \in D(S^*) |\ S^* f = \pm i f\} \\ &= \{f \in \mathcal{H} | (Sh, f) = \mp i(h, f) \ \ \forall\ h \in D(S)\} \\ &= \operatorname{im}(S \pm iI)^{\perp}.\end{aligned}$$

The dimensions of $\mathcal{L}^{\pm}$ (possibly infinite) are called the **deficiency indices** of S. Notice that the deficiency indices of an operator are the same as the deficiency indices of its closure.

An symmetric operator H is called **essentially self adjoint** if its closure is self-adjoint.

Theorem 1 *The following three conditions on a symmetric operator H are equivalent:*

1. *H is essentially self adjoint.*

2. *The deficiency indices of H are both 0.*

3. *H has exactly one self-adjoint extension.*

Furthermore, a symmetric operator has a self-adjoint extension if and only if its deficiency indices are equal, and then the set of self-adjoint extensions of S is in one to one correspondence with the set of all unitary isomorphisms of $\mathcal{L}^+$ with $\mathcal{S}^-$.

1. *H is essentially self adjoint.*

2. *The deficiency indices of H are both* 0.

Since the deficiency indices do not change when we pass to the closure, we may assume that H is closed. So condition 1) says that H is self-adjoint. Then if $f \in \mathcal{L}^+$ we have $Hf = H^*f = if$ which says that $i(f, f) = (Hf, f) = (f, Hf) = -i(f, f)$ so $f = 0$. This shows that 1) implies 2).

1. *H is essentially self adjoint.*

2. *The deficiency indices of H are both* 0.

The operator $(H + iI)$ maps $D(H)$ onto the subspace $\mathrm{im}(H+iI)$ with a bounded inverse. Since H is closed so is $(H+iI)^{-1}$ and hence $\mathrm{im}(H+iI)$ is a closed subspace of $\mathcal{H}$ whose orthogonal complement is 0. So $\mathrm{im}(H + iI) = \mathcal{H}$. So for any $f \in D(H^*)$ there is a $g \in D(H)$ such that $(H+iI)g = (H^*+iI)f$. This says that $(H^*+iI)(f-g) = 0$ so $f-g \in \mathcal{L}^- = \{0\}$. So $f = g$ and hence $f \in D(H)$. this show that $H = H^*$ so H is self-adjoint. This proves that 2) implies 1).

If the deficiency indices of H are both zero then H has no proper symmetric extension since $\mathrm{im}(H + iI) = \mathcal{H} = \mathrm{im}(H - iI)$. so $H^* = H$. This shows that 2) implies 3).

We now prove the last statement of the theorem and use it to prove that 3) implies 1). If K is a self adjoint extension of H, the deficiency indices of K must be zero by 2). The Cayley transform of K must map $\mathcal{L}^+$ unitarily onto $\mathcal{L}^-$ since these are the orthogonal complements of $\mathrm{im}\, H \pm iI$. So the deficiency indices must be equal. The set of self-adjoint extensions of H are then in one to one correspondence with the set of unitary isomorphisms of $\mathcal{L}^+$ with $\mathcal{L}^-$ which is the last assertion of the theorem. This set is infinite unless the deficiency indices are both zero which shows that 3) implies 2) and hence 1). $\square$

Consider $Hf = -if'$ defined on the space of smooth functions of compact support on

1. all of $\mathbb{R}$
2. on $(0, \infty)$ or
3. on $(0, 1)$.

In all cases, the condition $Hf = \pm if$ implies that $f' = \pm f$ in the sense of distributions and that $f, f' \in L_2$. So $f = ce^{\pm x}$. In case 1) this implies that $f = 0$ so the operator is essentially self adjoint. In case 2) we have $e^{-x} \in L_2$ while $e^{x} \notin L_2$ so the deficiency indices are unequal and H has no self adjoint extension. In case 3) both deficiency indices are equal to one and the set of self adjoint extensions are parametrized by the points on the unit circle.