Math 212b Lecture 7

Quadratic forms

Outline

Fractional powers of a non-negative self-adjoint operator.

Let H be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ with spectrum S. The spectral theorem tells us that there is a finite measure μ on $S \times \mathbb{N}$ and a unitary isomorphism

$$U : \mathcal{H} \to L_2 = L_2(S \times \mathbb{N}, \mu)$$

such that UHU^{-1} is multiplication by the function $h(s,n) = s$ and such that $\xi \in \mathcal{H}$ lies in $\mathrm{Dom}(H)$ if and only if $h \cdot (U\xi) \in L_2$.

Non-negative self- adjoint operators.

Clearly

$$(H\xi, \xi) \geq 0$$

for all $\xi \in \mathcal{H}$ if and only if μ assigns measure zero to the set $\{(s, n), s < 0\}$ in which case the spectrum of multiplication by h, which is the same as saying that the spectrum of H is contained in $[0, \infty)$. When this happens, we say that H is non-negative.

We say that $H \geq c$ if $H - cI$ is non-negative.

If H is non-negative, and $\lambda > 0$, we would like to define H^λ as being unitarily equivalent to multiplication by h^λ. As the spectral theorem does not say that the μ, L_2, and U are unique, so we have to check that this is well defined.

For this consider the function f on $\mathbb{R}$ defined by

$$f(x) = \frac{1}{|x|^\lambda + 1}.$$

f is a symbol and so by our functional calculus, $f(H)$ is well defined, and in any spectral representation goes over into multiplication by $f(h)$ which is injective. So $K = f(H)^{-1} - I$ is a well defined (in general unbounded) self-adjoint operator whose spectral representation is multiplication by h^λ. But the expression for K is independent of the spectral representation. This shows that $H^\lambda = K$ is well defined.

Proposition 1 *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $\mathrm{Dom}(H)$ be the domain of H. Let $0 < \lambda < 1$. Then $f \in \mathrm{Dom}(H)$ if and only if $f \in \mathrm{Dom}(H^\lambda)$ and $H^\lambda f \in \mathrm{Dom}(H^{1-\lambda})$ in which case*

$$Hf = H^{1-\lambda} H^\lambda f.$$

In particular, if $\lambda = \frac{1}{2}$, and we define $B_H(f,g)$ for $f, g \in \mathrm{Dom}(H^{\frac{1}{2}})$ by

$$B_H(f,g) := (H^{\frac{1}{2}} f, H^{\frac{1}{2}} g),$$

then $f \in \mathrm{Dom}(H)$ if and only if $f \in \mathrm{Dom}(H^{\frac{1}{2}})$ and also there exists a $k \in \mathcal{H}$ such that

$$B_H(f,g) = (k,g) \quad \forall\ g \in \mathrm{Dom}(H^{\frac{1}{2}})$$

in which case

$$Hf = k.$$

Proof. For the first part of the Proposition we may use the spectral representation: The Proposition then asserts that $f \in L_2$ satisfies $\int |h|^2 |f|^2 d\mu < \infty$ if and only if

$$\int (1+|h|^{2\lambda})|f|^2 d\mu < \infty \text{ and } \quad \int (1+|h|^{2(1-\lambda)})|h^\lambda f|^2 d\mu < \infty$$

which is obvious, as is the assertion that then $hf = h^{1-\lambda}(h^\lambda f)$.

The assertion that there exists a k such that $B_H(f,g) = (k,g) \quad \forall\, g \in \mathrm{Dom}(H^{\frac{1}{2}})$ is the same as saying that $H^{\frac{1}{2}} f \in \mathrm{Dom}((H^{\frac{1}{2}})^*)$ and $(H^{\frac{1}{2}})^* H^{\frac{1}{2}} f = k$. But $H^{\frac{1}{2}} = (H^{\frac{1}{2}})^*$ so the second part of the proposition follows from the first. $\square$

Quadratic forms.

The second half of Proposition 1 suggests that we study non-negative sesquilinear forms defined on some dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$. So we want to study

$$B: \ \mathcal{D} \times \mathcal{D} \to \mathbb{C}$$

such that

- $B(f,g)$ is linear in f for fixed g,
- $B(g,f) = \overline{B(f,g)}$, and
- $B(f,f) \geq 0$.

Of course, by the usual polarization trick such a B is determined by the corresponding **quadratic form**

$$Q(f) := B(f,f).$$

- $B(f, g)$ is linear in f for fixed g,
- $B(g, f) = \overline{B(f, g)}$, and
- $B(f, f) \geq 0$.

Of course, by the usual polarization trick such a B is determined by the corresponding **quadratic form**

$$Q(f) := B(f, f).$$

We would like to find conditions on B (or Q) which guarantee that $B = B_H$ for some non-negative self adjoint operator H as given by Proposition 1.

That *some* condition is necessary is exhibited by the following

Counterexample.

Let $\mathcal{H} = L_2(\mathbb{R})$ and let $\mathcal{D}$ consist of all continuous functions of compact support. Let

$$B(f, g) = f(0)\overline{g(0)}.$$

The only candidate for an operator H which satisfies $B(f, g) = (Hf, g)$ is the "operator" which consists of multiplication by the delta function at the origin. But there is no such operator.

Counterexample, continued.

Consider a sequence of uniformly bounded continuous functions f_n of compact support which are all identically one in some neighborhood of the origin and whose support shrinks to the origin. Then $f_n \to 0$ in the norm of $\mathcal{H}$. Also, $Q(f_n - f_m, f_n - f_m) \equiv 0$, so $Q(f_n - f_m, f_n - f_m) \to 0$. But $Q(f_n, f_n) \equiv 1 \neq 0 = Q(0, 0)$. So $\mathcal{D}$ is not complete for the norm $\| \cdot \|_1$

$$\|f\|_1 := (Q(f) + \|f\|_{\mathcal{H}}^2)^{\frac{1}{2}}.$$

Consider a function $g \in \mathcal{D}$ which equals one on the interval $[-1, 1]$ so that $(g, g) = 1$. Let $g_n := g - f_n$ with f_n as above. Then $g_n \to g$ in $\mathcal{H}$ yet $Q(g_n) \equiv 0$. So Q is *not* lower semi-continuous as a function on $\mathcal{D}$.

Lower semi-continuity.

Let X be a topological space, and let $Q : X \to \mathbb{R}$ be a real valued function. Let $x_0 \in X$. We say that Q is **lower semi-continuous** at x_0 if, for every $\epsilon > 0$ there is a neighborhood $U = U(x_0, \epsilon)$ of x_0 such that

$$Q(x) < Q(x_0) + \epsilon \quad \forall\, x \in U.$$

We say that Q is **lower semi-continuous** if it is lower semi-continuous at all points of X.

Proposition 2 *Let $\{Q_\alpha\}_{\alpha \in I}$ be a family of lower semi-continuous functions. Then*

$$Q := \sup_\alpha Q_\alpha$$

is lower semi-continuous. In particular, the pointwise limit of an increasing sequence of lower-semicontinuous functions is lower semi-continuous.

Proof. Let $x_0 \in X$ and $\epsilon > 0$. There exists an index α such that $Q_\alpha(x_0) > Q(x_0) - \frac{1}{2}\epsilon$. Then there exists a neighborhood U of x_0 such that $Q_\alpha(x) > Q_\alpha(x_0) - \frac{1}{2}\epsilon$ for all $x \in U$ and hence

$$Q(x) \geq Q_\alpha(x) > Q(x_0) - \epsilon \;\forall\; x \in U. \quad \square$$

It is easy to check that the sum and the inf of two lower semi-continuous functions is lower semi-continuous.

The main theorem about quadratic forms.

Let $\mathcal{H}$ be a separable Hilbert space and Q a non-negative quadratic form defined on a dense domain $\mathcal{D} \subset \mathcal{H}$. We

may extend the domain of definition of Q by setting it equal to $+\infty$ at all points of $\mathcal{H} \setminus \mathcal{D}$. Then we can say that the domain of Q consists of those f such that $Q(f) < \infty$. This will be a little convenient in the formulation of the next theorem.

Theorem 1 *The following conditions on Q are equivalent:*

1. *There is a non-negative self-adjoint operator H on $\mathcal{H}$ such that $\mathcal{D} = \mathrm{Dom}(H^{\frac{1}{2}})$ and*

$$Q(f) = \|H^{\frac{1}{2}} f\|^2.$$

2. *Q is lower semi-continuous as a function on $\mathcal{H}$.*
3. *$\mathcal{D} = \mathrm{Dom}(Q)$ is complete relative to the norm*

$$\|f\|_1 := \left(\|f\|^2 + Q(f)\right)^{\frac{1}{2}}.$$

1. implies 2. As H is non-negative, the operators $nI+H$ are invertible with bounded inverse, and $(nI+H)^{-1}$ maps $\mathcal{H}$ onto the domain of H. Consider the quadratic forms

$$Q_n(f) := \left(nH(nI+H)^{-1}f, f\right) = \left(H(I+n^{-1}H)^{-1}f, f\right)$$

which are bounded and continuous on all of $\mathcal{H}$. In the spectral representation of H, the space $\mathcal{H}$ is unitarily equivalent to $L_2(S,\mu)$ where $S = \mathrm{Spec}(H) \times \mathbb{N}$ and H goes over into multiplication by the function h where

$$h(s,k) = s.$$

The quadratic forms Q_n thus go over into the quadratic forms $\tilde{Q}_n$ where

$$\tilde{Q}_n(g) = \int \frac{nh}{n+h} g \cdot \overline{g} d\mu$$

The quadratic forms Q_n thus go over into the quadratic forms $\tilde{Q}_n$ where

$$\tilde{Q}_n(g) = \int \frac{nh}{n+h} g \cdot \overline{g} d\mu$$

for any $g \in L_2(S, \mu)$. The functions

$$\frac{nh}{n+h}$$

form an increasing sequence of functions on S, and hence the functions Q_n form an increasing sequence of continuous functions on $\mathcal{H}$. Hence their limit is lower semi-continuous. In the spectral representation, this limit is the quadratic form

$$g \mapsto \int hg \cdot \overline{g} d\mu$$

which is the spectral representation of the quadratic form Q.

2. implies 3. Let $\{f_n\}$ be a Cauchy sequence of elements of $\mathcal{D}$ relative to $\|\cdot\|_1$. Since $\|\cdot\| \leq \|\cdot\|_1$, $\{f_n\}$ is Cauchy with respect to the norm $\|\cdot\|$ of $\mathcal{H}$ and so converges in this norm to an element $f \in \mathcal{H}$. We must show that $f \in \mathcal{D}$ and that $f_n \to f$ in the $\|\cdot\|_1$ norm. Let $\epsilon > 0$. Choose N such that

$$\|f_m - f_n\|_1^2 = Q(f_m - f_n) + \|f_m - f_n\|^2 < \epsilon^2 \quad \forall\, m,n > N.$$

Let $m \to \infty$. By the lower semi-continuity of Q we conclude that

$$Q(f - f_n) + \|f - f_n\|^2 \leq \epsilon^2$$

and hence $f \in \mathcal{D}$ and $\|f - f_n\|_1 < \epsilon$. $\square$

3. implies 1. Let $\mathcal{H}_1$ denote the Hilbert space $\mathcal{D}$ equipped with the $\|\cdot\|_1$ norm. Notice that the scalar product on this Hilbert space is

$$(f,g)_1 = B(f,g) + (f,g)$$

where $B(f,f) = Q(f)$. The original scalar product $(\cdot,\cdot)$ is a bounded quadratic form on $\mathcal{H}_1$, so there is a bounded self-adjoint operator A on $\mathcal{H}_1$ such that $0 \leq A \leq 1$ and

$$(f,g) = (Af,g)_1 \quad \forall\, f,g \in \mathcal{H}_1.$$

$$(f,g) = (Af,g)_1 \quad \forall\ f,g \in \mathcal{H}_1.$$

Now apply the spectral theorem to A. So there is a unitary isomorphism U of $\mathcal{H}_1$ with $L_2(S,\mu)$ where $S = [0,1] \times \mathbb{N}$ such that UAU^{-1} is multiplication by the function a where $a(s,k) = s$. Since $(Af,f)_1 = 0 \Rightarrow f = 0$ we see that the set $\{0,k)\}$ has measure zero relative to μ so $a > 0$ except on a set of μ measure zero. So the function

$$h = a^{-1} - 1$$

is well defined and non-negative almost everywhere relative to μ. We have $a = (1+h)^{-1}$ and

$$(f,g) = \int_S \frac{1}{1+h} \hat{f}\overline{\hat{g}} d\mu$$

while

$$Q(f,g) + (f,g) = (f,g)_1 = \int_S f\overline{g} d\mu.$$

$$(f,g) = \int_S \frac{1}{1+h} \hat{f}\overline{\hat{g}} d\mu$$

while

$$Q(f,g) + (f,g) = (f,g)_1 = \int_S f\overline{g} d\mu.$$

Define the new measure ν on S by

$$\nu = \frac{1}{1+h}\mu.$$

Then the two previous equations imply that $\mathcal{H}$ is unitarily equivalent to $L_2(S,\nu)$, i.e.

$$(f,g) = \int_S f\overline{g} d\nu$$

and

$$Q(f,g) = \int_S hf\overline{g} d\nu.$$

$$(f, g) = \int_S f\overline{g}d\nu$$

and

$$Q(f, g) = \int_S hf\overline{g}d\nu.$$

This last equation says that Q is the quadratic form associated to the operator H corresponding to multiplication by h. $\square$

Extensions and cores.

A form Q satisfying the condition(s) of Theorem 1 is said to be **closed**. A form Q_2 is said to be an **extension** of a form Q_1 if it has a larger domain but coincides with Q_1 on the domain of Q_1. A form Q is said to be **closable** if it has a closed extension, and its smallest closed extension is called its **closure** and is denoted by $\overline{Q}$. If Q is closable, then the domain of $\overline{Q}$ is the completion of $\mathrm{Dom}(Q)$ relative to the metric $\| \cdot \|_1$ in Theorem 1. In general, we can consider this completion; but only for closable forms can we identify the completion as a subset of $\mathcal{H}$. A subset $\mathcal{D}$ of $\mathrm{Dom}(Q)$ where Q is closed is called a **core** of Q if Q is the completion of the restriction of Q to $\mathcal{D}$.

Proposition 3 *Let Q_1 and Q_2 be quadratic forms with the same dense domain $\mathcal{D}$ and suppose that there is a constant $c > 1$ such that*

$$c^{-1}Q_1(f) \le Q_2(f) \le cQ_1(f) \qquad \forall\ f \in \mathcal{D}.$$

If Q_1 is the form associated to a non-negative self-adjoint operator H_1 as in Theorem 1 then Q_2 is associated with a self-adjoint operator H_2 and

$$\mathrm{Dom}(H_1^{\frac{1}{2}}) = \mathrm{Dom}(H_2^{\frac{1}{2}}) = \mathcal{D}.$$

Proof. The assumption on the relation between the forms implies that their associated metrics on $\mathcal{D}$ are equivalent. So if $\mathcal{D}$ is complete with respect to one metric it is complete with respect to the other, and the domains of the associated self-adjoint operators both coincide with $\mathcal{D}$.

The Friedrichs extension of a symmetric operator.

Recall that an operator A defined on a dense domain $\mathcal{D}$ is called **symmetric** if

$$(Af,g) = (f,Ag) \quad \forall\ f,g \in \mathcal{D}.$$

A symmetric operator is called non-negative if

$$(Af,f) \geq 0$$

Theorem 2 [Friedrichs.] *Let Q be the form defined on the domain $\mathcal{D}$ of a symmetric operator A by*

$$Q(f) = (Af,f).$$

Then Q is closable and its closure is associated with a self-adjoint extension H of A.

Proof. Let $\mathcal{H}_1$ be the completion of $\mathcal{D}$ relative to the metric $\|\cdot\|_1$ as given in Theorem 1. The first step is to show that we can realize $\mathcal{H}_1$ as a subspace of $\mathcal{H}$. Since $\|f\| \leq \|f\|_1$, the identity map $f \mapsto f$ extends to a contraction $C : \mathcal{H}_1 \to \mathcal{H}$. We want to show that this map is injective. Suppose not, so that $Cf = 0$ for some $f \neq 0 \in \mathcal{H}_1$. Thus there exists a sequence $f_n \in \mathcal{D}$ such that

$$\|f - f_n\|_1 \to 0 \quad \text{and} \quad \|f_n\| \to 0.$$

So

$$\begin{aligned}
\|f\|_1^2 &= \lim_{m\to\infty} \lim_{n\to\infty} (f_m, f_n)_1 \\
&= \lim_{m\to\infty} \lim_{n\to\infty} \{(Af_m, f_n) + (f_m, f_n)\} \\
&= \lim_{m\to\infty} [(Af_m, 0) + (f_m, 0)] = 0.
\end{aligned}$$

So C is injective and hence Q is closable. Let H be the self-adjoint operator associated with the closure of Q. We must show that H is an extension of A. For $f, g \in \mathcal{D} \subset \mathrm{Dom}(H)$ we have

$$(H^{\frac{1}{2}} f, H^{\frac{1}{2}} g) = Q(f, g) = (Hf, g).$$

Since $\mathcal{D}$ is dense in $\mathcal{H}_1$, this holds for $f \in \mathcal{D}$ and $g \in \mathcal{H}_1$. By Proposition 1 this implies that $f \in \mathrm{Dom}(H)$. In other words, H is an extension of A. $\square$

Dirichlet boundary conditions.

In this section Ω will denote a bounded open set in $\mathbb{R}^N$, with piecewise smooth boundary, $c > 1$ is a constant, b is a continuous function defined on the closure $\overline{\Omega}$ of Ω satisfying

$$c^{-1} < b(x) < c \quad \forall\, x \in \overline{\Omega}$$

and

$$a = (a_{ij}) = (a_{ij}(x))$$

is a real symmetric matrix valued function of x defined and continuously differentiable on $\overline{\Omega}$ and satisfying

$$c^{-1} I \leq a(x) \leq cI \quad \forall\, x \in \overline{\Omega}.$$

Let

$$\mathcal{H}_b := L_2(\Omega, b d^N x).$$

An operator in divergence form.

Let

$$\mathcal{H}_b := L_2(\Omega, b d^N x).$$

We let $C^\infty(\overline{\Omega})$ denote the space of all functions f which are C^∞ on Ω and all of whose partial derivatives can be extended to be continuous functions on $\overline{\Omega}$. We let

$$C_0^\infty(\overline{\Omega}) \subset C^\infty(\overline{\Omega})$$

denote those f satisfying $f(x) = 0$ for $x \in \partial\Omega$.

For $f \in C_0^\infty(\overline{\Omega})$ we define Af by

$$Af(x) := -b(x)^{-1} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial f}{\partial x_j} \right).$$

The associated quadratic form.

$$Af(x) := -b(x)^{-1} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial f}{\partial x_j} \right).$$

Of course this operator is defined on $C^\infty(\overline{\Omega})$ but for $f, g \in C_0^\infty(\overline{\Omega})$ we have, by Gauss's theorem (integration by parts)

$$(Af, g)_b = -\int_\Omega \left(\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial f}{\partial x_j} \right) \right) \overline{g} d^N x$$

$$= \int_\Omega \sum_{ij} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial \overline{g}}{\partial x_j} d^N x = (f, Ag)_b.$$

So if we define the quadratic form

$$Q(f,g) := \int_\Omega \sum_{ij} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial \overline{g}}{\partial x_j} d^N x, \tag{1}$$

then Q is symmetric and so defines a quadratic form associated to the non-negative symmetric operator H. We may apply the Friedrichs theorem to conclude the existence of a self adjoint extension H of A which is associated to the closure of Q.

The closure of Q is complete relative to the metric determined by Theorem 1. But our assumptions about b and a guarantee the metrics of quadratic forms coming from different choices of b and a are equivalent and all equivalent to the metric coming from the choice $b \equiv 1$ and $a \equiv (\delta_{ij})$ which is

$$\|f\|_1^2 = \int_\Omega \left(|f|^2 + |\nabla f|^2\right) d^N x, \tag{2}$$

where

$$\nabla f = \partial_1 f \oplus \partial_2 f \oplus \cdots \oplus \partial_N f$$

and

$$|\nabla f|^2(x) = (\partial_1 f(x))^2 + \cdots + (\partial_1 f(x))^2 .$$

To compare this with Proposition 3, notice that now the Hilbert spaces $\mathcal{H}_b$ will also vary (but are equivalent in norm) as well as the metrics on the domain of the closure of Q.

The Sobolev spaces $W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$.

Let us be more explicit about the completion of $C^\infty(\overline{\Omega})$ and $C_0^\infty(\overline{\Omega})$ relative to this metric. If $f \in L_2(\Omega, d^N x)$ then f defines a linear function on the space of smooth functions of compact support contained in Ω by the usual rule

$$\ell_f(\phi) := \int_\Omega f\phi d^N x \quad \forall \ \phi \in C_c^\infty(\Omega).$$

We can then define the partial derivatives of f in the sense of the theory of distributions, for example

$$\ell_{\partial_i f}(\phi) = -\int_\Omega f(\partial_i \phi) d^N x.$$

These partial derivatives may or may not come from elements of $L_2(\Omega, d^N x)$.

These partial derivatives may or may not come from elements of $L_2(\Omega, d^N x)$. We define the space $W^{1,2}(\Omega)$ to consist of those $f \in L_2(\Omega, d^N x)$ whose first partial derivatives (in the distributional sense) $\partial_i f = \partial f / \partial x_i$ all come form elements of $L_2(\Omega, d^N x)$. We define a scalar product $(\ ,\ )_1$ on $W^{1,2}(\Omega)$ by

$$(f, g)_1 := \int_\Omega \left\{ f(x)\overline{g(x)} + \nabla f(x) \cdot \overline{\nabla g(x)} \right\} d^N x. \qquad (3)$$

$$(f,g)_1 := \int_\Omega \left\{ f(x)\overline{g(x)} + \nabla f(x) \cdot \overline{\nabla g(x)} \right\} d^N x. \qquad (3)$$

It is easy to check that $W^{1,2}(\Omega)$ is a Hilbert space, i.e. is complete. Indeed, if f_n is a Cauchy sequence for the corresponding metric $\|\|_1$, then f_n and the $\partial_i f_n$ are Cauchy relative to the metric of $L_2(\Omega, d^N x)$, and hence converge in this metric to limits, i.e.

$$f_n \to f \quad \text{and } \partial_i f_n \to g_i \qquad i = 1, \ldots N$$

for some elements f and $g_1, \ldots g_N$ of $L_2(\Omega, d^N x)$. We must show that $g_i = \partial_i f$. But for any $\phi \in C_c^\infty(\Omega)$ we have

$$\begin{aligned} \ell_{g_i}(\phi) = (g_i, \overline{\phi}) &= \lim_{n\to\infty} (\partial_i f_n, \overline{\phi}) \\ &= -\lim_{n\to\infty} (f_n, \partial_i \overline{\phi}) \\ &= -(f, \partial_i \overline{\phi}) \end{aligned}$$

which says that $g_i = \partial_i f$.

Lemma 1 *$C_c^\infty(\Omega)$ is dense in $C_0^\infty(\overline{\Omega})$ relative to the metric $\|\cdot\|_1$ given by (2).*

Proof. By taking real and imaginary parts, it is enough to prove this theorem for real valued functions. For any $\epsilon > 0$ let F_ϵ be a smooth real valued function on $\mathbb{R}$ such that

- $F_\epsilon(x) = x \quad \forall\ |x| > 2\epsilon$
- $F_\epsilon(x) = 0 \quad \forall |x| < \epsilon$
- $|F_\epsilon(x)| \leq |x| \quad \forall\ x \in \mathbb{R}$
- $0 \leq F'_\epsilon(x) \leq 3 \quad \forall x \in \mathbb{R}.$

For $f \in C_0^\infty(\overline{\Omega})$ define

$$f_\epsilon(x) := F_\epsilon(f(x)),$$

$$f_\epsilon(x) := F_\epsilon(f(x)),$$

so $F_\epsilon \in C_c^\infty(\Omega)$. Also,

$$|f_\epsilon(x)| \leq |f(x)| \quad \text{and} \quad \lim_{\epsilon \to 0} f_\epsilon(x) = f(x) \; \forall \; x \in \Omega.$$

So the dominated convergence theorem implies that $\|f - f_\epsilon\|_2 \to 0$. We have to establish convergence in L_2 of the derivatives.

Consider the set $B \subset \Omega$ where $f = 0$ and $\nabla(f) \neq 0$. By the implicit function theorem, this is a union of hypersurfaces, and so has measure zero. We have

$$\int_\Omega |\nabla(f) - \nabla(f_\epsilon)|^2 d^N x = \int_{\Omega \setminus B} |\nabla(f) - \nabla(f_\epsilon)|^2 d^N x.$$

On all of Ω we have $|\partial_i(f_\epsilon)| \leq 3|\partial_i f|$ and on $\Omega \setminus B$ we have $\partial_i f_\epsilon(x) \to \partial_i f(x)$. So the dominated convergence theorem proves the L_2 convergence of the partial derivatives. □

As a consequence, we see that the domain of $\overline{Q}$ is precisely $W_0^{1,2}(\Omega)$.

Generalizing the domain and the coefficients.

Let Ω be any open subset of $\mathbb{R}^n$, let b be any measurable function defined on Ω and satisfying

$$c^{-1} < b(x) < c \quad \forall\, x \in \Omega$$

for some $c > 1$ and a a measurable matrix valued function defined on Ω and satisfying

$$c^{-1} I \leq (x) \leq cI \quad \forall\, x \in \Omega.$$

We can still define the Hilbert space

$$\mathcal{H}_b := L_2(\Omega, bd^N x)$$

as before, but can not define the operator A as above.

Nevertheless we can define the closed form

$$\overline{Q}(f) = \int_\Omega \sum_{ij} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial \overline{g}}{\partial x_j} d^N x,$$

on $W_0^{1,2}(\Omega)$ which we know to be closed because the metric it determines by Theorem 1 is equivalent as a metric to the norm on $W_0^{1,2}(\Omega)$. Therefore, by Theorem 1, there is a non-negative self-adjoint operator H such that

$$(H^{\frac{1}{2}} f, H^{\frac{1}{2}} g)_b = Q(f, g) \quad \forall\ f, g \in W_0^{1,2}(\Omega).$$

A Sobolev version of Rademacher's theorem.

Recall that Rademacher's theorem says that a Lipschitz function on $\mathbb{R}^N$ is differentiable almost everywhere with a bound on its derivative given by the Lipschitz constant. The following is a variant of this theorem which is useful for our purposes.

Theorem 3 *Let f be a continuous real valued function on $\mathbb{R}^N$ which vanishes outside a bounded open set Ω and which satisfies*

$$|f(x) - f(y)| \leq c||x - y|| \quad \forall\ x, y\ \in \mathbb{R}^N \tag{4}$$

for some $c < \infty$. Then $f \in W_0^{1,2}(\Omega)$.

We break the proof up into several steps:

Proposition 4 *Suppose that f satisfies (4) and the support of f is contained in a compact set K. Then*

$$f \in W^{1,2}(\mathbb{R}^N)$$

and

$$\|f\|_1^2 = \int_{\mathbb{R}^N} \left(|f|^2 + |\nabla f|^2\right) d^N x \leq |K| c^2 (N + \operatorname{diam}(K))$$

where $|K|$ denotes the Lebesgue measure of K.

Mollification.

Proof. Let k be a C^∞ function on $\mathbb{R}^N$ such that

- $k(x) = 0$ if $\|x\| \geq 1$,
- $k(x) > 0$ if $\|x\| < 1$, and
- $\int_{\mathbb{R}^N} k(x) d^N x = 1$.

Define k_s by

$$k_s(x) = s^{-N} k\left(\frac{x}{s}\right).$$

So

- $k_s(x) = 0$ if $\|x\| \geq s$,
- $k_s(x) > 0$ if $\|x\| < s$, and
- $\int_{\mathbb{R}^N} k_s(x) d^N x = 1$.

- $k_s(x) = 0$ if $\|x\| \geq s$,
- $k_s(x) > 0$ if $\|x\| < s$, and
- $\int_{\mathbb{R}^N} k_s(x) d^N x = 1$.

Define p_s by

$$p_s(x) := \int_{\mathbb{R}^N} k_s(x - z) f(z) d^N y$$

so p_s is smooth,

$$\operatorname{supp} p_s \subset K_s = \{x|\ d(x,K) \leq s\}$$

and

$$p_s(x) - p_s(y) = \int_{\mathbb{R}^N} (f(x-z) - f(y-z))\, k_s(z) d^N z$$

so

$$|p_s(x) - p_s(y)| \leq c\|x-y\|.$$

This implies that $\|\nabla p_s(x)\| \leq c$ so the mean value theorem implies that $\sup_{x\in\mathbb{R}^N} |p_s(x)| \leq c \cdot \operatorname{diam} K_s$ and so

$$\|p_s\|_1^2 \leq |K_s| c^2 (\operatorname{diam} K_s^2 + N).$$

By Plancherel

$$\|p_s\|_1^2 = \int_{\mathbb{R}^N} (1 + \|\xi\|^2)|\hat{p}_s(\xi)|^2 d^N\xi$$

and since convolution goes over into multipication

$$\hat{p}_s(\xi) = \hat{f}(\xi)h(s\xi)$$

where

$$h(\xi) = \int_{\mathbb{R}^N} k(x)e^{-ix\cdot\xi} d^N x.$$

The function h is smooth with $h(0) = 1$ and $|h(\xi)| \leq 1$ for all ξ.

$$\hat{p}_s(\xi) = \hat{f}(\xi)h(s\xi)$$

where

$$h(\xi) = \int_{\mathbb{R}^N} k(x)e^{-ix\cdot\xi}d^N x.$$

The function h is smooth with $h(0) = 1$ and $|h(\xi)| \leq 1$ for all ξ. By Fatou's lemma

$$\begin{aligned}
\|f\|_1 &= \int_{\mathbb{R}^N} (1 + \|\xi\|^2)|\hat{f}(\xi)|^2 d^N\xi \\
&\leq \liminf_{s\to 0} \int_{\mathbb{R}^N} (1 + \|\xi\|^2)|h(sy)|^2|\hat{f}(\xi)|^2 d^N\xi \\
&= \liminf_{s\to 0} \|p_s\|_1^2 \\
&\leq |K|c^2(N + \mathrm{diam}(K)). \quad \square
\end{aligned}$$

The dominated convergence theorem implies that

$$\|f - p_s\|_1^2 \to 0$$

as $s \to 0$. But the support of p_s is slightly larger than the support of f, so we are not able to conclude directly that $f \in W_0^{1,2}(\Omega)$. So we first must cut f down to zero where it is small. We do this by defining the real valued functions ϕ_ϵ on $\mathbb{R}$ by

$$\phi_\epsilon(s) = \begin{cases} 0 & \text{if} & |s| \le \epsilon \\ s & \text{if} & |s| \ge 2\epsilon \\ 2(s-\epsilon) & \text{if} & \epsilon \le s \le 2\epsilon \\ 2(s+\epsilon) & \text{if} & -2\epsilon \le s \le -\epsilon \end{cases} .$$

$$\phi_\epsilon(s) = \begin{cases} 0 & \text{if} \quad |s| \leq \epsilon \\ s & \text{if} \quad |s| \geq 2\epsilon \\ 2(s-\epsilon) & \text{if} \quad \epsilon \leq s \leq 2\epsilon \\ 2(s+\epsilon) & \text{if} \quad -2\epsilon \leq s \leq -\epsilon \end{cases}.$$

Then set $f_\epsilon = \phi_\epsilon(f)$. If O is the open set where $f(x) \neq 0$ then f_ϵ has its support contained in the set S_ϵ consisting of all points whose distance from the complement of O is $> \epsilon/c$. Also

$$|f_\epsilon(x) - f_\epsilon(y)| \leq 2|f(x) - f(y)| \leq 2\|x - y|.$$

So we may apply the preceding result to f_ϵ to conclude that $f_\epsilon \in W^{1,2}(\mathbb{R}^N)$ and

$$\|f\|_1^2 \leq 4|S_\epsilon|c^2(N + \text{diam } (O)^2)$$

So we may apply the preceding result to f_ϵ to conclude that $f_\epsilon \in W^{1,2}(\mathbb{R}^N)$ and

$$\|f\|_1^2 \leq 4|S_\epsilon|c^2(N + \text{diam }(O)^2)$$

and then by Fatou applied to the Fourier transforms as before that

$$\|f\|_1^2 \leq 4|O|c^2(N + \text{diam }(O)).$$

Also, for ϵ sufficiently small $f_\epsilon \in W_0^{1,2}(\Omega)$. So we will be done if we show that $\|f_\epsilon - f\| \to 0$ as $\epsilon \to 0$. The set L_ϵ on which this difference is $\neq 0$ is contained in the set of all x for which $0 < |f(x)| < 2\epsilon$ which decreases to the empty set as $\epsilon \to 0$. The above argument shows that

$$\|f - f_\epsilon\|_1 \leq 4|L_\epsilon c^2(N + \text{diam }(L_\epsilon)^2) \to 0. \quad \square$$