Math 212b lecture 8

The first 16 slides were downloaded from Davies: spectral theory - classification of generators.

Theorem 3.56 *A closed, densely defined operator Z acting in the Banach space $\mathcal{B}$ is the generator of a one-parameter semigroup T_t satisfying*

$$\|T_t\| \leq M\mathrm{e}^{at} \tag{3.17}$$

for all $t \geq 0$ if and only if

$$\mathrm{Spec}(Z) \subseteq \{z : \mathrm{Re}\,(z) \leq a\}$$

and

$$\|(\lambda I - Z)^{-m}\| \leq M(\lambda - a)^{-m} \tag{3.18}$$

for all $\lambda > a$ and all $m \geq 1$.

$$\|T_t\| \leq M\mathrm{e}^{at} \tag{3.17}$$

Proof If T_t satisfies (3.17) and $\lambda > a$ and $m \geq 1$ then

$$\begin{aligned}
\|(\lambda I - Z)^{-m}\| &= \left\| \int_0^\infty \ldots \int_0^\infty T_{t_1+\ldots+t_m} \mathrm{e}^{-\lambda(t_1+\ldots+t_m)} \, \mathrm{d}t_1 \ldots \mathrm{d}t_m \right\| \\
&\leq \int_0^\infty \ldots \int_0^\infty M\mathrm{e}^{-(\lambda-a)(t_1+\ldots+t_m)} \, \mathrm{d}t_1 \ldots \mathrm{d}t_m \\
&= M(\lambda - a)^{-m}.
\end{aligned}$$

The converse is much harder since we have to construct the semigroup T_t. The idea is to approximate Z by bounded operators Z_λ and show that the semigroups

$$T_t^\lambda = \mathrm{e}^{Z_\lambda t}$$

converge as $\lambda \to +\infty$ to a semigroup T_t whose generator is Z.

If $\lambda > a$ we define the bounded operator Z_λ by

$$\begin{aligned} Z_\lambda &= \lambda Z(\lambda I - Z)^{-1} \\ &= -\lambda\{1 - \lambda(\lambda I - Z)^{-1}\}. \end{aligned}$$

We first show that

$$\lim_{\lambda \to \infty} \|Z(\lambda I - Z)^{-1} f\| = 0 \tag{3.19}$$

for all $f \in \mathcal{B}$. Because

$$\begin{aligned} \|Z(\lambda I - Z)^{-1}\| &= \|1 - \lambda(\lambda I - Z)^{-1}\| \\ &\leq 1 + \frac{M\lambda}{\lambda - a} \end{aligned}$$

is bounded as $\lambda \to \infty$, it is sufficient to prove this for f in a dense subset of $\mathcal{B}$. If $f \in \mathrm{Dom}(Z)$ then

$$\begin{aligned} \|Z(\lambda I - Z)^{-1} f\| &\leq \|(\lambda I - Z)^{-1}\| \, \|Zf\| \\ &\leq \frac{M\|Zf\|}{\lambda - a} \end{aligned}$$

and this converges to 0 as $\lambda \to \infty$.

We next show that

$$\lim_{\lambda\to\infty} Z_\lambda f = Zf \tag{3.20}$$

for all $f \in \mathrm{Dom}(Z)$. For any such f and any $b > a$ there exists $g \in \mathcal{B}$ such that $f = (bI - Z)^{-1}g$. Hence

$$\begin{aligned}
&\lim_{\lambda\to\infty} \|Z_\lambda f - Zf\| \\
&= \lim_{\lambda\to\infty} \|\lambda Z(\lambda I - Z)^{-1}(bI - Z)^{-1}g - Z(bI - Z)^{-1}g\| \\
&= \lim_{\lambda\to\infty} \|\lambda Z\{(\lambda I - Z)^{-1} - (bI - Z)^{-1}\}(b-\lambda)^{-1}g - Z(bI - Z)^{-1}g\| \\
&= \lim_{\lambda\to\infty} \left\|\left(\frac{\lambda}{\lambda - b} - 1\right) Z(bI - Z)^{-1}g - \frac{\lambda}{\lambda - b} Z(\lambda I - Z)^{-1}g\right\| \\
&\leq \lim_{\lambda\to\infty} \frac{b}{\lambda - b}\|Z(bI - Z)^{-1}g\| + \lim_{\lambda\to\infty} \frac{\lambda}{\lambda - b}\|Z(\lambda I - Z)^{-1}g\| \\
&= 0
\end{aligned}$$

by (3.19).

If T_t^λ is the norm continuous semigroup defined by

$$T_t^\lambda = \sum_{n=0}^{\infty} t^n Z_\lambda^n / n!$$

then

$$\begin{aligned} \|T_t^\lambda\| &= \left\| \sum_{n=0}^{\infty} t^n \{-\lambda + \lambda^2 (\lambda I - Z)^{-1}\}^n / n! \right\| \\ &= \left\| \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} t^n \lambda^{2n} (\lambda I - Z)^{-n} / n! \right\| \\ &\leq \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} t^n \lambda^{2n} \|(\lambda I - Z)^{-n}\| / n! \\ &\leq \mathrm{e}^{-\lambda t} M \mathrm{e}^{t\lambda^2/(\lambda - a)} \\ &= M \mathrm{e}^{ta\lambda/(\lambda - a)} \\ &\leq M \mathrm{e}^{2at} \end{aligned} \tag{3.21}$$

provided $\lambda \geq 2a$. Moreover

$$\limsup_{\lambda \to \infty} \|T_t^\lambda\| \leq M \mathrm{e}^{at}. \tag{3.22}$$

$$\lim_{\lambda\to\infty} Z_\lambda f = Zf \tag{3.20}$$

$$\|T_t\| \leq Me^{2at} \tag{3.21}$$

$$\limsup_{\lambda\to\infty} \|T_t^\lambda\| \leq Me^{at}. \tag{3.22}$$

We next show that if $f \in \mathcal{B}$ then $T_t^\lambda f$ converges as $\lambda \to \infty$ uniformly for t in bounded intervals. By (3.21) it is sufficient to prove this when f lies in the dense set $\mathrm{Dom}(Z)$. For such f

$$\begin{aligned} \left\| \frac{\mathrm{d}}{\mathrm{d}s}\{T_{t-s}^\lambda T_s^\mu f\} \right\| &= \|T_{t-s}^\lambda(-Z_\lambda + Z_\mu)T_s^\mu f\| \\ &= \|T_{t-s}^\lambda T_s^\mu(-Z_\lambda + Z_\mu)f\| \\ &\leq M^2 e^{4at}\|(-Z_\lambda + Z_\mu)f\|. \end{aligned}$$

Integrating with respect to s for $0 \leq s \leq t$ we obtain

$$\|T_t^\lambda f - T_t^\mu f\| \leq tM^2 e^{4at}\|(-Z_\lambda + Z_\mu)f\|,$$

which converges to zero as $\lambda, \mu \to \infty$, uniformly for t in bounded intervals by (3.20).

$$\limsup_{\lambda\to\infty} \|T_t^\lambda\| \le M\mathrm{e}^{at}. \tag{3.22}$$

It is an immediate consequence of (3.22) and the semigroup properties of T_t^λ that $T_0 = 1$, $\|T_t\| \le M\mathrm{e}^{at}$ for all $t \ge 0$ and $T_sT_t = T_{s+t}$ for all $s,t \ge 0$. The uniformity of the convergence for t in bounded intervals implies that T_tf is jointly continuous in t and f, and so is a one-parameter semigroup.

Our final task is to verify that the generator B of T_t coincides with Z. We start from the equation

$$T_t^\lambda f - f = \int_0^t T_x^\lambda Z_\lambda f \,\mathrm{d}x \tag{3.23}$$

valid for all $f \in \mathcal{B}$ by Lemma 2.2. If $f \in \mathrm{Dom}(Z)$ then we let $\lambda \to \infty$ in (3.23) and use (3.20) to obtain

$$T_tf - f = \int_0^t T_x Z f \,\mathrm{d}x.$$

Dividing by t and letting $t \to 0$ we see that $f \in \mathrm{Dom}(B)$ and $Bf = Zf$. Hence Z is an extension of B. Since both $(\lambda I - Z)$ and $(\lambda I - B)$ are one-one with range equal to $\mathcal{B}$ for all $\lambda > a$ it follows that $Z = B$. ☐

Contraction semi-groups.

A one-parameter *contraction semigroup* is defined
as a one-parameter semigroup
such that $\|T_t\| \leq 1$ for all $t \geq 0$.

The Hille Yosida theorem.

Corollary 3.57 *(Hille-Yosida Theorem) The closed, densely defined operator Z acting in the Banach space $\mathcal{B}$ is the generator of a one-parameter contraction semi-group if and only if*

$$\mathrm{Spec}(Z) \subseteq \{z : \mathrm{Re}\,(z) \le 0\}$$

and

$$\|(\lambda I - Z)^{-1}\| \le \lambda^{-1} \tag{3.24}$$

for all $\lambda > 0$.

Proof Put $M = 1$ and $a = 0$ in Theorem 3.56. □

Dissipative operators.

We next reformulate Corollary 3.57 directly in terms of the operator Z. If Z acts in $\mathcal{B}$ with domain $\mathcal{D}$, we let $\mathcal{E}$ denote the set of pairs $(f, \phi) \in \mathcal{B} \times \mathcal{B}^*$ such that $f \in \mathcal{D}$, $\|f\| = 1$, $\|\phi\| = 1$ and $\langle f, \phi \rangle = 1$. Note that for each such f a suitable ϕ exists by the Hahn-Banach theorem, but it need not be unique. We say that Z is *dissipative* if $\mathrm{Re}\,(\langle f, \phi \rangle) \leq 0$ for all $(f, \phi) \in \mathcal{E}$. If Z is an operator with domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$ then Z is dissipative if and only if $\mathrm{Re}\,(\langle Zf, f \rangle) \leq 0$ for all $f \in \mathcal{D}$.

The Lumer Phillips theorem.

Theorem 3.59 *(Lumer-Phillips Theorem) An operator Z with dense domain $\mathcal{D}$ in a Banach space $\mathcal{B}$ is the generator of a one-parameter contraction semigroup if and only if it is dissipative and the range of $(\lambda I - Z)$ equals $\mathcal{B}$ for all $\lambda > 0$.*

Proof If Z satisfies the second set of conditions and $(f, \phi) \in \mathcal{E}$ then

$$\begin{aligned} \|(\lambda I - Z)f\| &\geq |\langle(\lambda I - Z)f, \phi\rangle| \\ &= |\lambda - \langle Zf, \phi\rangle| \\ &\geq \lambda \\ &= \lambda\|f\|. \end{aligned}$$

Therefore the operator $(\lambda I - Z)$ is one-one with range equal to $\mathcal{B}$, and

$$\|(\lambda I - Z)^{-1}\| \leq \lambda^{-1}.$$

We may now apply Corollary 3.57.

Conversely suppose Z is the generator of a one-parameter contraction semi-group T_t. If $(f, \phi) \in \mathcal{E}$ then

$$\begin{aligned} \text{Re } \langle Zf, \phi \rangle &= \text{Re } \lim_{h \to 0} h^{-1} \langle T_h f - f, \phi \rangle \\ &= \text{Re } \lim_{h \to 0} h^{-1} \{ \langle T_h f, \phi \rangle - 1 \} \\ &\leq \lim_{h \to 0} h^{-1} \{ \|T_h\| \, \|f\| \, \|\phi\| - 1 \} \\ &\leq 0. \end{aligned}$$

□

Theorem 3.60 *Let Z be a closable operator with dense domain $\mathcal{D}$ in a Banach space $\mathcal{B}$, and suppose that the range of $(\lambda I - Z)$ is dense for some λ satisfying* $\mathrm{Re}\,(\lambda) > 0$. *Suppose also that for all $f \in \mathcal{D}$ there exists $\phi \in \mathcal{B}^*$ such that $\|\phi\| = 1$, $\quad \langle f, \phi \rangle = 1$ and $\mathrm{Re}\,(\langle Zf, \phi\rangle) \leq 0$.*

Then Z is dissipative and the closure of Z is the generator of a one-parameter contraction semigroup. Moreover

$$\|(\lambda I - Z)^{-1}\| \leq \mathrm{Re}\,(\lambda)^{-1}$$

for all λ satisfying $\mathrm{Re}\,(\lambda) > 0$.

Let Y be the closure of Z.

Proof As before we see that

$$\|(\mu I - Z)f\| \geq \mu \|f\|$$

for all $\mu > 0$ and all $f \in \mathcal{D}$. Therefore

$$\|(\mu I - Y)f\| \geq \mu \|f\|$$

for all $\mu > 0$ and all $f \in \mathrm{Dom}(Y)$.

This implies that $(\lambda I - Y)$ has range

equal to $\mathcal{B}$ and that

$$\|(\lambda I - Y)^{-1}\| \leq \lambda^{-1}.$$

Hence

$$\mathrm{Spec}(Y) \cap \{z : |z - \lambda| < \lambda\} = \emptyset$$

and that $(\mu I - Y)$ has range equal to $\mathcal{B}$ for all μ such that $0 < \mu < 2\lambda$. Replacing λ by $3\lambda/2$ in the above argument, it follows by induction that

$$\mathrm{Spec}(Y) \subseteq \{z : \mathrm{Re}\,(z) \leq 0\}.$$

The proof is now completed by applying Corollary 3.57. □

I now go to my notes and repeat the Hille - Yosida theorem and the Lumer - Phillips theorem.

In finite dimensions we have the formula

$$e^{tB} = \sum_0^\infty \frac{t^k}{k!} B^k$$

with convergence guaranteed as a result of the convergence of the usual exponential series in one variable. (There are serious problems with this definition from the point of view of numerical implementation which we will not discuss here.)

In infinite dimensional spaces some additional assumptions have to be placed on an operator B before we can conclude that the above series converges. Here is a very stringent condition which nevertheless suffices for our purposes.

Let $\mathbf{F}$ be a Frechet space and B a continuous map of $\mathbf{F} \to \mathbf{F}$. We will assume that the B^k are **equibounded** in the sense that for any defining semi-norm p there is a constant K and a defining semi-norm q such that

$$p(B^k x) \leq K q(x) \quad \forall\ k = 1, 2, \ldots \quad \forall\ x \in \mathbf{F}.$$

Here the K and q are required to be independent of k and x.

Then

$$p(\sum_m^n \frac{t^k}{k!} B^k x) \leq \sum_m^n \frac{t^k}{k!} p(B^k x) \leq K q(x) \sum_n^n \frac{t^k}{k!}$$

$$p(\sum_m^n \frac{t^k}{k!} B^k x) \le \sum_m^n \frac{t^k}{k!} p(B^k x) \le K q(x) \sum_n^n \frac{t^k}{k!}$$

and so

$$\sum_0^n \frac{t^k}{k!} B^k x$$

is a Cauchy sequence for each fixed t and x (and uniformly in any compact interval of t). It therefore converges to a limit. We will denote the map $x \mapsto \sum_0^\infty \frac{t^k}{k!} B^k x$ by

$$\exp(tB).$$

This map is linear, and the computation above shows that

$$p(\exp(tB)x) \le K \exp(t) q(x).$$

The usual proof (using the binomial formula) shows that $t \mapsto \exp(tB)$ is a one parameter equibounded semi-group. More generally, if B and C are two such operators then if $BC = CB$ then $\exp(t(B+C)) = (\exp tB)(\exp tC)$.

Also, from the power series it follows that the infinitesimal generator of $\exp tB$ is B.

Let us now return to the general case of an equibounded semigroup T_t with infinitesimal generator A on a Frechet space $\mathbf{F}$ where we know that the resolvent $R(z, A)$ for Re $z > 0$ is given by

$$R(z, A)x = \int_0^\infty e^{-zt} T_t x dt.$$

This formula shows that $R(z, A)x$ is continuous in z. The resolvent equation

$$R(z, A) - R(w, A) = (w - z)R(z, A)R(w, A)$$

then shows that $R(z, A)x$ is complex differentiable in z with derivative $-R(z, A)^2 x$. It then follows that $R(z, A)x$ has complex derivatives of all orders given by

$$\frac{d^n R(z, A)x}{dz^n} = (-1)^n n! R(z, A)^{n+1} x.$$

$$R(z, A)x = \int_0^\infty e^{-zt} T_t x dt.$$

$$\frac{d^n R(z, A)x}{dz^n} = (-1)^n n! R(z, A)^{n+1} x.$$

On the other hand, differentiating the integral formula for the resolvent n- times gives

$$\frac{d^n R(z, A)x}{dz^n} = \int_0^\infty e^{-zt} (-t)^n T_t dt$$

where differentiation under the integral sign is justified by the fact that the T_t are equicontinuous in t. Putting the previous two equations together gives

$$(zR(z, A))^{n+1} x = \frac{z^{n+1}}{n!} \int_0^\infty e^{-zt} t^n T_t x dt.$$

$$(zR(z,A))^{n+1}x = \frac{z^{n+1}}{n!}\int_0^\infty e^{-zt}t^n T_t x dt.$$

This implies that for any semi-norm p we have

$$p((zR(z,A))^{n+1}x) \leq \frac{z^{n+1}}{n!}\int_0^\infty e^{-zt}t^n \sup_{t\geq 0} p(T_t x) dt$$

$$= \sup_{t\geq 0} p(T_t x)$$

since

$$\int_0^\infty e^{-zt}t^n dt = \frac{n!}{z^{n+1}}.$$

Since the T_t are equibounded by hypothesis, we conclude

Proposition 1 *The family of operators $\{(zR(z,A))^n\}$ is equibounded in Re $z > 0$ and $n = 0, 1, 2, \ldots$.*

The Hille - Yosida theorem.

Theorem 5 [**Hille-Yosida.**] *Let* A *be an operator with dense domain* $D(A)$*, and such that the resolvents*

$$R(n, A) = (nI - A)^{-1}$$

exist and are bounded operators for $n = 1, 2, \ldots$. *Then* A *is the infinitesimal generator of a uniquely determined equibounded semigroup if and only if the operators*

$$\{(I - n^{-1}A)^{-m}\}$$

are equibounded in $m = 0, 1, 2 \ldots$ *and* $n = 1, 2, \ldots$.

Proof. If A is the infinitesimal generator of an equibounded semi-group then we know that the $\{(I-n^{-1}A)^{-m}\}$ are equibounded by virtue of the preceding proposition. So we must prove the converse.

Set

$$J_n = (I - n^{-1}A)^{-1}$$

so $J_n = n(nI - A)^{-1}$ and so for $x \in D(A)$ we have

$$J_n(nI - A)x = nx$$

or

$$J_n Ax = n(J_n - I)x.$$

Similarly $(nI - A)J_n = nI$ so $AJ_n = n(J_n - I)$. Thus we have

$$AJ_n x = J_n Ax = n(J_n - I)x \quad \forall\, x \in D(A). \tag{14}$$

$$AJ_n x = J_n A x = n(J_n - I)x \quad \forall\ x \in D(A). \tag{14}$$

The idea of the proof is now this: By the results of the preceding section, we can construct the one parameter semigroup $s \mapsto \exp(sJ_n)$. Set $s = nt$. We can then form $e^{-nt}\exp(ntJ_n)$ which we can write as $\exp(tn(J_n - I)) = \exp(tAJ_n)$ by virtue of (14). We expect from (5) that

$$\lim_{n\to\infty} J_n x = x \quad \forall \quad x \in \mathbf{F}. \tag{15}$$

This then suggests that the limit of the $\exp(tAJ_n)$ be the desired semi-group.

$$\lim_{s\to\infty} sR(s)x = x \quad \forall\ x \in \mathbf{F}. \tag{5}$$

$$\lim_{n\to\infty} J_n x = x \quad \forall \quad x \in \mathbf{F}. \tag{15}$$

So we begin by proving (15). We first prove it for $x \in D(A)$. For such x we have $(J_n - I)x = n^{-1} J_n A x$ by (14) and this approaches zero since the J_n are equibounded. But since $D(A)$ is dense in $\mathbf{F}$ and the J_n are equibounded we conclude that (15) holds for all $x \in \mathbf{F}$.

Now define

$$T_t^{(n)} = \exp(tAJ_n) := \exp(nt(J_n - I)) = e^{-nt}\exp(ntJ_n).$$

$$T_t^{(n)} = \exp(tAJ_n) := \exp(nt(J_n - I)) = e^{-nt}\exp(ntJ_n).$$

We know from the preceding section that

$$p(\exp(ntJ_n)x) \leq \sum \frac{(nt)^k}{k!} p(J_n^k x) \leq e^{nt} K q(x)$$

which implies that

$$p(T_t^{(n)} x) \leq Kq(x). \tag{16}$$

Thus the family of operators $\{T_t^{(n)}\}$ is equibounded for all $t \geq 0$ and $n = 1, 2, \ldots$. We next want to prove that the $\{T_t^{(n)}\}$ converge as $n \to \infty$ uniformly on each compact interval of t:

The J_n commute with one another by their definition, and hence J_n commutes with $T_t^{(m)}$. By the semi-group property we have

$$\frac{d}{dt}T_t^m x = AJ_m T_t^{(m)} x = T_t^{(m)} AJ_m x$$

so

$$T_t^{(n)}x - T_t^{(m)}x = \int_0^t \frac{d}{ds}(T_{t-s}^{(m)}T_s^{(n)})x ds =$$

$$\int_0^t T_{t-s}^{(m)}(AJ_n - AJ_m)T_s^{(n)} x ds.$$

Applying the semi-norm p and using the equiboundedness we see that

$$p(T_t^{(n)}x - T_t^{(m)}x) \leq Ktq((J_n - J_m)Ax).$$

From (15) this implies that the $T_t^{(n)}x$ converge (uniformly in every compact interval of t) for $x \in D(A)$, and hence since $D(A)$ is dense and the $T_t^{(n)}$ are equicontinuous for all $x \in \mathbf{F}$. The limiting family of operators T_t are equicontinuous and form a semi-group because the $T_t^{(n)}$ have this property.

$$p(T_t^{(n)}x) \leq Kq(x). \qquad (16)$$

We must show that the infinitesimal generator of this semi-group is A. Let us temporarily denote the infinitesimal generator of this semi-group by B, so that we want to prove that $A = B$. Let $x \in D(A)$. We claim that

$$\lim_{n \to \infty} T_t^{(n)} AJ_n x = T_t Ax \qquad (17)$$

uniformly in in any compact interval of t. Indeed, for any semi-norm p we have $p(T_t Ax - T_t^{(n)} AJ_n x)$

$$\begin{aligned} &\leq \quad p(T_t Ax - T_t^{(n)} Ax) + p(T_t^{(n)} Ax - T_t^{(n)} AJ_n x) \\ &\leq \quad p((T_t - T_t^{(n)})Ax) + Kq(Ax - J_n Ax) \end{aligned}$$

where we have used (16) to get from the second line to the third.

$$p(T_t Ax - T_t^{(n)} AJ_n x)$$

$$\leq \quad p((T_t - T_t^{(n)})Ax) + Kq(Ax - J_n Ax)$$

where we have used (16) to get from the second line to the third. The second term on the right tends to zero as $n \to \infty$ and we have already proved that the first term converges to zero uniformly on every compact interval of t. This establishes (17).

$$\lim_{n\to\infty} T_t^{(n)} AJ_n x = T_t Ax \tag{17}$$

$$\lim_{n\to\infty} T_t^{(n)} A J_n x = T_t A x \tag{17}$$

We thus have, for $x \in D(A)$,

$$\begin{aligned} T_t x - x &= \lim_{n\to\infty} (T_t^{(n)} x - x) \\ &= \lim_{n\to\infty} \int_0^t T_s^{(n)} A J_n x ds \\ &= \int_0^t (\lim_{n\to\infty} T_s^{(n)} A J_n x) ds \\ &= \int_0^t T_s A x ds \end{aligned}$$

where the passage of the limit under the integral sign is justified by the uniform convergence in t on compact sets.

It follows from $T_t x - x = \int_0^t T_s Ax ds$ that x is in the domain of the infinitesimal operator B of T_t and that $Bx = Ax$. So B is an extension of A in the sense that $D(B) \supset D(A)$ and $Bx = Ax$ on $D(A)$.

But since B is the infinitesimal generator of an equi-bounded semi-group, we know that $(I - B)$ maps $D(B)$ onto $\mathbf{F}$ bijectively, and we are assuming that $(I - A)$ maps $D(A)$ onto $\mathbf{F}$ bijectively. Hence $D(A) = D(B)$. QED

In case $\mathbf{F}$ is a Banach space, so there is a single norm $p = \| \ \|$, the hypotheses of the theorem read: $D(A)$ is dense in $\mathbf{F}$, the resolvents $R(n, A)$ exist for all integers $n = 1, 2, \ldots$ and there is a constant K independent of n and m such that

$$\|(I - n^{-1}A)^{-m}\| \leq K \ \ \forall\, n = 1, 2, \ldots, \ m = 1, 2, \ldots. \tag{18}$$

In particular, if A satisfies

$$\|(I - n^{-1}A)^{-1}\| \leq 1 \tag{19}$$

condition (18) is satisfied, and such an A then generates a semi-group. Under this stronger hypothesis we can draw a stronger conclusion: In (16) we now have $p = q = \| \cdot \|$ and $K = 1$. Since $\lim_{n\to\infty} T_t^n x = T_t x$ we see that under the hypothesis (19) we can conclude that

$$\|T_t\| \leq 1 \quad \forall\, t \geq 0.$$

A semi-group T_t satisfying this condition is called a **contraction semi-group**. We will study another useful condition for recognizing a contraction semigroup in the following subsection.

Let **F** be a Banach space. Recall that a semi-group T_t is called a **contraction semi-group** if

$$\|T_t\| \leq 1 \quad \forall\, t \geq 0,$$

and that (19) is a sufficient condition on operator with dense domain to generate a contraction semi-group.

The Lumer-Phillips theorem to be stated below gives a necessary and sufficient condition on the infinitesimal generator of a semi-group for the semi-group to be a contraction semi-group. It is generalization of the fact that the resolvent of a self-adjoint operator has $\pm i$ in its resolvent set.

The first step is to introduce a sort of fake scalar product in the Banach space $\mathbf{F}$. A **semi-scalar product** on $\mathbf{F}$ is a rule which assigns a number $\langle\langle x, z\rangle\rangle$ to every pair of elements $x, z \in \mathbf{F}$ in such a way that

$$\begin{aligned} \langle\langle x+y, z\rangle\rangle &= \langle\langle x, z\rangle\rangle + \langle\langle y, z\rangle\rangle \\ \langle\langle \lambda x, z\rangle\rangle &= \lambda\langle\langle x, z\rangle\rangle \\ \langle\langle x, x\rangle\rangle &= \|x\|^2 \\ |\langle\langle x, z\rangle\rangle| &\leq \|x\| \cdot \|z\|. \end{aligned}$$

We can always choose a semi-scalar product as follows: by the Hahn-Banach theorem, for each $z \in \mathbf{F}$ we can find an $\ell_z \in \mathbf{F}^*$ such that

$$\|\ell_z\| = \|z\| \text{ and } \ell_z(z) = \|z\|^2.$$

Choose one such ℓ_z for each $z \in \mathbf{F}$ and set $\langle\langle x, z\rangle\rangle := \ell_z(x)$.

An operator A with domain $D(A)$ on $\mathbf{F}$ is called **dissipative** relative to a given semi-scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ if

$$\mathrm{Re}\ \langle\langle Ax, x \rangle\rangle \leq 0 \quad \forall\ x \in D(A).$$

For example, if A is a symmetric operator on a Hilbert space such that

$$(Ax, x) \leq 0 \ \ \forall\ x \in D(A) \tag{20}$$

then A is dissipative relative to the scalar product.

Theorem 6 [Lumer-Phillips.] *Let A be an operator on a Banach space $\mathbf{F}$ with $D(A)$ dense in $\mathbf{F}$. Then A generates a contraction semi-group if and only if A is dissipative with respect to any semi-scalar product and*

$$\mathrm{Range}(I - A) = \mathbf{F}.$$

Proof. Suppose first that $D(A)$ is dense and that $\mathrm{Range}(I - A) = \mathbf{F}$. We wish to show that (19) holds, which will guarantee that A generates a contraction semi-group. Let $s > 0$. Then if $x \in D(A)$ and $y = sx - Ax$ then

$$s\|x\|^2 = s\langle\langle x, x\rangle\rangle \leq s\langle\langle x, x\rangle\rangle - \mathrm{Re}\ \langle\langle Ax, x\rangle\rangle = \mathrm{Re}\ \langle\langle y, x\rangle\rangle$$

implying

$$s\|x\|^2 \leq \|y\|\|x\|. \tag{21}$$

$$s\|x\|^2 \leq \|y\|\|x\|. \tag{21}$$

We are assuming that Range$(I - A) = \mathbf{F}$. This together with (21) with $s = 1$ implies that $R(1, A)$ exists and

$$\|R(1, A)\| \leq 1.$$

In turn, this implies that for all z with $|z - 1| < 1$ the resolvent $R(z, A)$ exists and is given by the power series

$$R(z, A) = \sum_{n=0}^{\infty} (z-1)^n R(1, A)^{n+1}$$

by our general power series formula for the resolvent.

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by our general power series formula for the resolvent. In particular, for s real and $|s-1| < 1$ the resolvent exists, and then (21) implies that $\|R(s,A)\| \le s^{-1}$. Repeating the process we keep enlarging the resolvent set $\rho(A)$ until it includes the whole positive real axis and conclude from (21) that $\|R(s,A)\| \le s^{-1}$ which implies (19). As we are assuming that $D(A)$ is dense we conclude that A generates a contraction semigroup.

$$\|(I - n^{-1}A)^{-1}\| \le 1 \qquad (19)$$

Conversely, suppose that T_t is a contraction semi-group with infinitesimal generator A. We know that $\mathrm{Dom}(A)$ is dense. Let $\langle\langle\cdot,\cdot\rangle\rangle$ be any semi-scalar product. Then

$$\mathrm{Re}\ \langle\langle T_t x - x, x\rangle\rangle = \mathrm{Re}\ \langle\langle T_t x, x\rangle\rangle - \|x\|^2 \leq \|T_t x\|\|x\| - \|x\|^2 \leq 0.$$

Dividing by t and letting $t \searrow 0$ we conclude that $\mathrm{Re}\ \langle\langle Ax, x\rangle\rangle \leq 0$ for all $x \in D(A)$, i.e. A is dissipative for $\langle\langle\cdot,\cdot\rangle\rangle$. QED

Once again, this gives a direct proof of the existence of the unitary group generated generated by a skew adjoint operator.

A useful way of verifying the condition $\text{Range}(I - A) = \mathbf{F}$ is the following: Let $A^* : \mathbf{F}^* \to \mathbf{F}^*$ be the adjoint operator which is defined if we assume that $D(A)$ is dense.

Proposition 2 *Suppose that A is densely defined and closed, and suppose that both A and A^* are dissipative. Then* $\text{Range}(I - A) = \mathbf{F}$ *and hence A generates a contraction semigroup.*

$$y = sx - Ax$$

$$s\|x\|^2 = s\langle\langle x, x\rangle\rangle \leq s\langle\langle x, x\rangle\rangle - \text{Re } \langle\langle Ax, x\rangle\rangle = \text{Re } \langle\langle y, x\rangle\rangle$$

implying

$$s\|x\|^2 \leq \|y\|\|x\|. \tag{21}$$

Proof. The fact that A is closed implies that $(I-A)^{-1}$ is closed, and since we know that $(I-A)^{-1}$ is bounded from the fact that A is dissipative, we conclude that Range$(I-A)$ is a closed subspace of F. If it were not the whole space there would be an $\ell \in F^*$ which vanished on this subspace, i.e.

$$\langle \ell, x - Ax\rangle = 0 \quad \forall\ x \in D(A).$$

This implies that that $\ell \in D(A^*)$ and $A^*\ell = \ell$ which can not happen if A^* is dissipative by (21) applied to A^* and $s = 1$. QED

A special case: $\exp(t(B-I))$ **with** $\|B\| \leq 1$.

Suppose that $B : F \to F$ is a bounded operator on a Banach space with $\|B\| \leq 1$. Then for any semi-scalar product we have

$$\text{Re } \langle\langle (B-I)x, x\rangle\rangle = \text{Re } \langle\langle Bx, x\rangle\rangle - \|x\|^2 \leq \|Bx\|\|x\| - \|x\|^2 \leq 0$$

so $B - I$ is dissipative and hence $\exp(t(B - I))$ exists as a contraction semi-group by the Lumer-Phillips theorem. We can prove this directly since we can write

$$\exp(t(B - I)) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k B^k}{k!}.$$

The series converges in the uniform norm and we have

$$\|\exp(t(B - I))\| \leq e^{-t} \sum \frac{t^k \|B\|^k}{k!} \leq 1.$$

For future use (Chernoff's theorem and the Trotter product formula) we record (and prove) the following inequality:

$$\|[\exp(n(B-I))-B^n]x\| \leq \sqrt{n}\|(B-I)x\| \quad \forall\, x \in \mathbf{F}, \text{ and } \forall\, n \tag{22}$$

Proof.

$$\begin{aligned}
&\|[\exp(n(B-I)) - B^n]x\| \\
&= \quad \|e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!}(B^k - B^n)x\| \\
&\leq \quad e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!}\|(B^k - B^n)x\| \\
&\leq \quad e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!}\|(B^{|k-n|} - I)x\|
\end{aligned}$$

$$e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}\|(B^{|k-n|}-I)x\|$$

$$\begin{aligned} &= e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}\|(B-I)(I+B+\cdots+B^{(|k-n|-1})x\| \\ &\leq e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}|k-n|\|(B-I)x\|. \end{aligned}$$

So to prove (22) it is enough establish the inequality

$$e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}|k-n|\leq\sqrt{n}. \tag{23}$$

Consider the space of all sequences $\mathbf{a} = \{a_0, a_1, \dots\}$ with finite norm relative to scalar product

$$(\mathbf{a}, \mathbf{b}) := e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!} a_k \overline{b_k}.$$

The Cauchy-Schwarz inequality applied to $\mathbf{a}$ with $a_k = |k-n|$ and $\mathbf{b}$ with $b_k \equiv 1$ gives

$$e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!} |k-n| \leq \sqrt{e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!} (k-n)^2} \cdot \sqrt{e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!}}.$$

The second square root is one, and we recognize the sum under the first square root as the variance of the Poisson distribution with parameter n, and we know that this variance is n. QED