Math 212b Lecture 9

Convergence of semi-groups and the Trotter product formula.

Lie's formula for the exponential of a sum.

Let A and B be linear operators on a finite dimensional Hilbert space. Lie's formula says that

$$\exp(A+B) = \lim_{n\to\infty} \left[(\exp A/n)(\exp B/n)\right]^n . \qquad (1)$$

Proof. Let $S_n := \exp(\frac{1}{n}(A+B))$ so that

$$S_n^n = \exp(A+B).$$

Let $T_n = (\exp A/n)(\exp B/n)$. We wish to show that

$$S_n^n - T_n^n \to 0.$$

Notice that the constant and the linear terms in the power series expansions for S_n and T_n are the same, so

$$\|S_n - T_n\| \leq \frac{C}{n^2}$$

where $C = C(A, B)$. We have the telescoping sum

$$S_n^n - T_n^n = \sum_{k=0}^{n-1} S_n^{k-1}(S_n - T_n)T^{n-1-k}$$

so

$$\|S_n^n - T_n^n\| \leq n\|S_n - T_n\| \left(\max(\|S_n\|, \|T_n\|)\right)^{n-1}.$$

But

$$\|S_n\| \leq \exp\frac{1}{n}(\|A\|+\|B\|) \quad \text{and} \quad \|T_n\| \exp\frac{1}{n}(\|A\|+\|B\|)$$

and $$\left[\exp\frac{1}{n}(\|A\|+\|B\|)\right]^{n-1} = \exp\frac{n-1}{n}(\|A\|+\|B\|)$$
$$\leq \exp(\|A\|+\|B\|)$$

so

$$\|S_n^n - T_n^n\| \leq \frac{C}{n}\exp(\|A\|+\|B\|. \quad \square$$

This same proof works if A and B are self-adjoint operators such that $A+B$ is self-adjoint on the intersection of their domains. For a proof see Reed-Simon vol. I pages 295-296. For applications this is too restrictive. So we give a more general formulation and proof following Chernoff.

Convergence of semigroups.

We are going to be interested in the following type of result. We would like to know that if A_n is a sequence of operators generating equibounded one parameter semi-groups $\exp tA_n$ and $A_n \to A$ where A generates an equi-bounded semi-group $\exp tA$ then the semi-groups converge, i.e. $\exp tA_n \to \exp tA$. We will prove such a result for the case of contractions. But before we can even formulate the result, we have to deal with the fact that each A_n comes equipped with its own domain of definition, $D(A_n)$. We do not want to make the overly restrictive hypothesis that these all coincide, since in many important applications they won't.

Cores (again).

For this purpose we make the following definition. Let us assume that $\mathbf{F}$ is a Banach space and that A is an operator on $\mathbf{F}$ defined on a domain $D(A)$. We say that a linear subspace $\mathbf{D} \subset D(A)$ is a **core** for A if the closure $\overline{A}$ of A and the closure of A restricted to $\mathbf{D}$ are the same: $\overline{A} = \overline{A|\mathbf{D}}$. This certainly implies that $D(A)$ is contained in the closure of $A|\mathbf{D}$. In the cases of interest to us $D(A)$ is dense in $\mathbf{F}$, so that every core of A is dense in $\mathbf{F}$.

We begin with an important preliminary result:

Proposition 3 *Suppose that A_n and A are dissipative operators, i.e. generators of contraction semi-groups. Let* $\mathbf{D}$ *be a core of A. Suppose that for each $x \in \mathbf{D}$ we have that $x \in D(A_n)$ for sufficiently large n (depending on x) and that*

$$A_n x \to Ax. \tag{24}$$

Then for any z with $\mathrm{Re}\, z > 0$ *and for all $y \in \mathbf{F}$*

$$R(z, A_n)y \to R(z, A)y. \tag{25}$$

Proof. We know that the $R(z, A_n)$ and $R(z, A)$ are all bounded in norm by $1/\mathrm{Re}\ z$. So it is enough for us to prove convergence on a dense set. Since $(zI - A)D(A) = \mathbf{F}$, it follows that $(zI - A)\mathbf{D}$ is dense in $\mathbf{F}$ since A is closed. So in proving (25) we may assume that $y = (zI - A)x$ with $x \in \mathbf{D}$. Then $\|R(z, A_n)y - R(z, A)y\| =$

$$
\begin{aligned}
&= \|R(z, A_n)(zI - A)x - x\| \\
&= \|R(z, A_n)(zI - A_n)x + R(z, A_n)(A_n x - Ax) - x\| \\
&= \|R(z, A_n)(A_n - A)x\| \\
&\leq \frac{1}{\mathrm{Re}\ z}\|(A_n - A)x\| \to 0,
\end{aligned}
$$

where, in passing from the first line to the second we are assuming that n is chosen sufficiently large that $x \in D(A_n)$. QED

Theorem 7 *Under the hypotheses of the preceding proposition,*

$$(\exp(tA_n))x \to (\exp(tA))x$$

for each $x \in \mathbf{F}$ uniformly on every compact interval of t.

Proof. Let

$$\phi_n(t) := e^{-t}[((\exp(tA_n))x - (\exp(tA))x)] \quad \text{for } t \geq 0$$

and set $\phi(t) = 0$ for $t < 0$. It will be enough to prove that these $\mathbf{F}$ valued functions converge uniformly in t to 0, and since $\mathbf{D}$ is dense and since the operators entering into the definition of ϕ_n are uniformly bounded in n, it is enough to prove this convergence for $x \in \mathbf{D}$ which is dense. We claim that for fixed $x \in \mathbf{D}$ the functions $\phi_n(t)$ are uniformly equi-continuous.

$$\phi_n(t) := e^{-t}[((\exp(tA_n))x - (\exp(tA))x)] \quad \text{for } t \geq 0$$

We claim that for fixed $x \in \mathbf{D}$ the functions $\phi_n(t)$ are uniformly equi-continuous. To see this observe that

$$\frac{d}{dt}\phi_n(t) =$$

$$e^{-t}[(\exp(tA_n))A_n x - (\exp(tA))Ax] - e^{-t}[(\exp(tA_n))x - (\exp(tA))x]$$

for $t \geq 0$ and the right hand side is uniformly bounded in $t \geq 0$ and n.

So to prove that $\phi_n(t)$ converges uniformly in t to 0, it is enough to prove this fact for the convolution $\phi_n \star \rho$ where ρ is any smooth function of compact support, since we can choose the ρ to have small support and integral $\sqrt{2\pi}$, and then $\phi_n(t)$ is close to $(\phi_n \star \rho)(t)$.

Now the Fourier transform of $\phi_n \star \rho$ is the product of their Fourier transforms: $\hat{\phi_n}\hat{\rho}$. We have $\hat{\phi_n}(s) =$

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(-1-is)t}[(\exp tA_n)x - (\exp(tA))x]dt =$$

$$\frac{1}{\sqrt{2\pi}}[R(1+is, A_n)x - R(1+is, A)x].$$

Thus by the proposition

$$\hat{\phi}_n(s) \to 0,$$

in fact uniformly in s. Hence using the Fourier inversion formula and, say, the dominated convergence theorem (for Banach space valued functions),

$$(\phi_n \star \rho)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\phi}_n(s)\hat{\rho}(s)e^{ist}ds \to 0$$

uniformly in t. QED

Chernoff's theorem.

In what follows, $\mathbf{F}$ is a Banach space. Eventually we will restrict attention to a Hilbert space.

Theorem 9 [Chernoff.] *Let $f : [0, \infty) \to$ bounded operators on $\mathbf{F}$ be a continuous map with*

$$\|f(t)\| \leq 1 \quad \forall\ t$$

and

$$f(0) = I.$$

Let A be a dissipative operator and $\exp tA$ the contraction semi-group it generates. Let $\mathbf{D}$ be a core of A. Suppose that

$$\lim_{h \searrow 0} \frac{1}{h}[f(h) - I]x = Ax \quad \forall\ x \in \mathbf{D}.$$

$$\|f(t)\| \leq 1 \quad \forall\, t \qquad\qquad f(0) = I.$$

Let A be a dissipative operator and $\exp tA$ *the contraction semi-group it generates. Let* $\mathbf{D}$ *be a core of A. Suppose that*

$$\lim_{h \searrow 0} \frac{1}{h}[f(h) - I]x = Ax \quad \forall\, x \in \mathbf{D}.$$

Then for all $y \in \mathbf{F}$

$$\lim \left[f\left(\frac{t}{n}\right)\right]^n y = (\exp tA)y \tag{26}$$

uniformly in any compact interval of $t \geq 0$.

Proof. For fixed $t > 0$ let

$$C_n := \frac{n}{t}\left[f\left(\frac{t}{n}\right) - I\right].$$

Then $\frac{t}{n}C_n$ generates a contraction semi-group by the special case of the Lumer-Phillips theorem discussed in Section 6.2, and therefore (by change of variable), so does C_n. So C_n is the generator of a semi-group

$$\exp tC_n$$

and the hypothesis of the theorem is that $C_n x \to Ax$ for $x \in \mathbf{D}$. Hence by the limit theorem in the preceding section

$$(\exp tC_n)y \to (\exp tA)y$$

for each $y \in \mathbf{F}$ uniformly on any compact interval of t.

$$\|[\exp(n(B-I))-B^n]x\| \leq \sqrt{n}\|(B-I)x\| \quad \forall\, x \in \mathbf{F}, \text{ and } \forall\, n \tag{22}$$

$$(\exp tC_n)y \to (\exp tA)y$$

for each $y \in \mathbf{F}$ uniformly on any compact interval of t. Now

$$\exp(tC_n) = \exp n\left[f\left(\frac{t}{n}\right) - I\right]$$

so we may apply (22) to conclude that

$$\|\left(\exp(tC_n) - f\left(\frac{t}{n}\right)^n\right)x\| \leq \sqrt{n}\|\left(f\left(\frac{t}{n}\right) - I\right)x\|$$

$$= \frac{t}{\sqrt{n}}\|\frac{n}{t}\left(f\left(\frac{t}{n}\right) - I\right)x\|.$$

$$\| \left(\exp(tC_n) - f\left(\frac{t}{n}\right)^n \right) x \| \leq \quad \frac{t}{\sqrt{n}} \| \frac{n}{t} \left(f\left(\frac{t}{n}\right) - I \right) x \|.$$

The expression inside the $\| \cdot \|$ on the right tends to Ax so the whole expression tends to zero. This proves (26) for all x in $\mathbf{D}$. But since $\mathbf{D}$ is dense in $\mathbf{F}$ and $f(t/n)$ and $\exp tA$ are bounded in norm by 1 it follows that (26) holds for all $y \in \mathbf{F}$. QED

$$\lim \left[f\left(\frac{t}{n}\right) \right]^n y = (\exp tA) y \qquad (26)$$

The product formula.

Let A and B be the infinitesimal generators of the contraction semi-groups $P_t = \exp tA$ and $Q_t = \exp tB$ on the Banach space F. Then $A+B$ is only defined on $D(A) \cap D(B)$ and in general we know nothing about this intersection. However let us *assume* that $D(A) \cap D(B)$ is sufficiently large that the closure $\overline{A+B}$ is a densely defined operator and $\overline{A+B}$ is in fact the generator of a contraction semi-group R_t. So $\mathbf{D} := D(A) \cap D(B)$ is a core for $\overline{A+B}$.

Theorem 10 [Trotter.] *Under the above hypotheses*

$$R_t y = \lim \left(P_{\frac{t}{n}} Q_{\frac{t}{n}} \right)^n y \quad \forall\ y \in \mathbf{F} \tag{27}$$

uniformly on any compact interval of $t \geq 0$.

Theorem 10 [Trotter.] *Under the above hypotheses*

$$R_t y = \lim \left(P_{\frac{t}{n}} Q_{\frac{t}{n}} \right)^n y \quad \forall\ y \in \mathbf{F} \tag{27}$$

uniformly on any compact interval of $t \geq 0$.

Proof. Define

$$f(t) = P_t Q_t.$$

For $x \in D$ we have

$$f(t)x = P_t(I + tB + o(t))x = (I + At + Bt + o(t))x$$

so the hypotheses of Chernoff's theorem are satisfied. The conclusion of Chernoff's theorem asserts (27). QED

A symmetric operator on a Hilbert space is called **essentially self adjoint** if its closure is self-adjoint. So a reformulation of the preceding theorem in the case of self-adjoint operators on a Hilbert space says

Theorem 11 *Suppose that S and T are self-adjoint operators on a Hilbert space H and suppose that $S+T$ (defined on $D(S)\cap D(T)$) is essentially self-adjoint. Then for every $y \in H$*

$$\exp(it((\overline{S+T}))y = \lim_{n\to\infty} \left(\exp(\frac{t}{n}iA)(\exp\frac{t}{n}iB)\right)^n y \tag{28}$$

where the convergence is uniform on any compact interval of t.

Commutators.

An operator A on a Hilbert space is called skew-symmetric if $A^* = -A$ on $D(A)$. This is the same as saying that iA is symmetric. So we call an operator skew adjoint if iA is self-adjoint. We call an operator A **essentially skew adjoint** if iA is essentially self-adjoint.

If A and B are bounded skew adjoint operators then their Lie bracket

$$[A, B] := AB - BA$$

is well defined and again skew adjoint.

In general, we can only define the Lie bracket on $D(AB) \cap D(BA)$ so we again must make some rather stringent hypotheses in stating the following theorem.

Theorem 12 *Let A and B be skew adjoint operators on a Hilbert space H and let*

$$\mathbf{D} := D(A^2) \cap D(B^2) \cap D(AB) \cap D(BA).$$

Suppose that the restriction of $[A, B]$ to $\mathbf{D}$ is essentially skew-adjoint. Then for every $y \in \mathbf{H}$

$$\exp t\overline{[A, B]}y =$$

$$\lim_{n\to\infty} \left((\exp -\sqrt{\frac{t}{n}}A)(\exp -\sqrt{\frac{t}{n}}B)(\exp \sqrt{\frac{t}{n}}A)(\exp \sqrt{\frac{t}{n}}B) \right)^n y \tag{29}$$

uniformly in any compact interval of $t \geq 0$.

Proof. The restriction of $[A, B]$ to **D** is assumed to be essentially skew-adjoint, so $[A, B]$ itself (which has the same closure) is also essentially skew adjoint.

We have

$$\exp(tA)x = (I + tA + \frac{t^2}{2}A^2)x + o(t^2)$$

for $x \in D$ with similar formulas for $\exp(-tA)$ etc.

Let

$$f(t) := (\exp -tA)(\exp -tB)(\exp tA)(\exp tB).$$

Multiplying out $f(t)x$ for $x \in D$ gives a whole lot of cancellations and yields

$$f(s)x = (I + s^2[A, B])x + o(s^2)$$

so (29) is a consequence of Chernoff's theorem with $s = \sqrt{t}$. QED

Recall: The Kato - Rellich theorem.

Theorem 13 [Kato-Rellich.] *Let A be a self-adjoint operator and B a symmetric operator with*

$$D(B) \supset D(A)$$

and

$$\|Bx\| \leq a\|Ax\| + b\|x\| \quad 0 \leq a < 1, \quad \forall \; x \in D(A).$$

Then $A + B$ is self-adjoint, and is essentially self-adjoint on any core of A.

Feynman path integrals.

Consider the operator

$$H_0 : \quad L_2(\mathbf{R}^3) \to L_2(\mathbf{R}^3)$$

given by

$$H_0 := -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right).$$

Here the domain of H_0 is taken to be those $\phi \in L_2(\mathbf{R}^3)$ for which the differential operator on the right, taken in the distributional sense, when applied to ϕ gives an element of $L_2(\mathbf{R}^3)$.

The operator H_0 is called the "free Hamiltonian of non-relativistic quantum mechanics". The Fourier transform $\mathcal{F}$ is a unitary isomorphism of $L_2(\mathbf{R}^3)$ into $L_2(\mathbf{R}_3)$ and carries H_0 into multiplication by ξ^2 whose domain consists of those $\hat{\phi} \in L_2(\mathbf{R}_3)$ such that $\xi^2\hat{\phi}(\xi)$ belongs to $L_2(\mathbf{R}_3)$. The operator consisting of multiplication by $e^{-it\xi^2}$ is clearly unitary, and provides us with a unitary one parameter group. Transferring this one parameter group back to $L_2(\mathbf{R}^3)$ via the Fourier transform gives us a one parameter group of unitary transformations whose infinitesimal generator is $-iH_0$.

Now the Fourier transform carries multiplication into convolution, and the inverse Fourier transform (in the distributional sense) of $e^{-i\xi^2 t}$ is $(2it)^{-3/2}e^{ix^2/4t}$. Hence we can write, in a formal sense,

$$(\exp(-itH_0)f)(x) = (4\pi it)^{-3/2}\int_{\mathbf{R}^3}\exp\left(\frac{i(x-y)^2}{4t}\right)f(y)dy.$$

Here the right hand side is to be understood as a long winded way of writing the left hand side which is well defined as a mathematical object. The right hand side can also be regarded as an actual integral for certain classes of f, and as the L_2 limit of such such integrals. We shall discuss this interpretation in Section 10.

Let V be a function on $\mathbf{R}^3$. We denote the operator on $L_2(\mathbf{R}^3)$ consisting of multiplication by V also by V. Suppose that V is such that $H_0 + V$ is again self-adjoint. For example, if V were continuous and of compact support this would certainly be the case by the Kato-Rellich theorem. (Realistic "potentials" V will not be of compact support or be bounded, but nevertheless in many important cases the Kato-Rellich theorem does apply.)

Then the Trotter product formula says that

$$\exp -it(H_0 + V) = \lim_{n\to\infty}\left(\exp(-i\frac{t}{n}H_0)(\exp -i\frac{t}{n}V)\right)^n .$$

We have

$$\left((\exp -i\frac{t}{n}V)f\right)(x) = e^{-i\frac{t}{n}V(x)}f(x).$$

Hence we can write the expression under the limit sign in the Trotter product formula, when applied to f and evaluated at x_0 as the following formal expression:

$$\left(\frac{4\pi it}{n}\right)^{-3n/2} \int_{\mathbf{R}^3} \cdots \int_{\mathbf{R}^3} \exp(iS_n(x_0,\ldots,x_n))f(x_n)dx_n \cdots dx_1$$

where

$$S_n(x_0,x_1,\ldots,x_n,t) := \sum_{i=1}^{n} \frac{t}{n}\left[\frac{1}{4}\left(\frac{(x_i - x_{i-1})}{t/n}\right)^2 - V(x_i)\right].$$

$$S_n(x_0, x_1, \ldots, x_n, t) := \sum_{i=1}^{n} \frac{t}{n} \left[\frac{1}{4} \left(\frac{(x_i - x_{i-1})}{t/n} \right)^2 - V(x_i) \right].$$

If $X : s \mapsto X(s),\ 0 \leq s \leq t$ is a piecewise differentiable curve, then the **action** of a particle of mass m moving along this curve is defined in classical mechanics as

$$S(X) := \int_0^t \left(\frac{m}{2} \dot{X}(s)^2 - V(X(s)) \right) ds$$

where $\dot{X}$ is the velocity (defined at all but finitely many points).

$$S_n(x_0, x_1, \ldots, x_n, t) := \sum_{i=1}^{n} \frac{t}{n}\left[\frac{1}{4}\left(\frac{(x_i - x_{i-1})}{t/n}\right)^2 - V(x_i)\right].$$

$$S(X) := \int_0^t \left(\frac{m}{2}\dot{X}(s)^2 - V(X(s))\right) ds$$

where $\dot{X}$ is the velocity (defined at all but finitely many points).

Take $m = \frac{1}{2}$ and let X be the polygonal path which goes from x_0 to x_1, from x_1 to x_2 etc., each in time t/n so that the velocity is $|x_i - x_{i-1}|/(t/n)$ on the i-th segment. Also, the integral of $V(X(s))$ over this segment is approximately $\frac{t}{n}V(x_i)$. The formal expression written above for the Trotter product formula can be thought of as an integral over polygonal paths (with step length t/n) of $e^{iS_n(X)}f(X(t))d_nX$ where S_n approximates the classical action and where d_nX is a measure on this space of polygonal paths.

This suggests that an intuitive way of thinking about the Trotter product formula in this context is to imagine that there is some kind of "measure" dX on the space Ω_{x_0} of *all* continuous paths emanating from x_0 and such that

$$\exp(-it(H_0 + V)f)(x) = \int_{\Omega_{x_0}} e^{iS(X)} f(X(t))dX.$$

This formula was suggested in 1942 by Feynman in his thesis (Trotter's paper was in 1959), and has been the basis of an enormous number of important calculations in physics, many of which have given rise to exciting mathematical theorems which were then proved by other means. I am unaware of any general mathematical justification of these "path integral" methods in the form that they are used.

The Feynman-Kac formula.

An important advance was introduced by Mark Kac in 1951 where the unitary group $\exp -it(H_0 + V)$ is replaced by the contraction semi-group $\exp -t(H_0 + V)$. Then the techniques of probability theory (in particular the existence of Wiener measure on the space of continuous paths) can be brought to bear to justify a formula for the contractive semi-group as an integral over path space. I will state and prove an elementary version of this formula which follows directly from what we have done. The assumptions about the potential are physically unrealistic, but I choose to regard the extension to a more realistic potential as a technical issue rather than a conceptual one.

Let V be a continuous real valued function of compact support. To each continuous path ω on $\mathbf{R}^n$ and for each fixed time $t \geq 0$ we can consider the integral

$$\int_0^t V(\omega(s))ds.$$

The map

$$\omega \mapsto \int_0^t V(\omega(s))ds \tag{30}$$

is a continuous function on the space of continuous paths, and we have

$$\frac{t}{m}\sum_{j=1}^{m} V\left(\omega\left(\frac{jt}{m}\right)\right) \to \int_0^t V(\omega(s))ds \tag{31}$$

for each fixed ω.

Theorem 14 The Feynman-Kac formula. *Let V be a continuous real valued function of compact support on $\mathbf{R}^n$. Let*

$$H = \Delta + V$$

as an operator on $\mathbf{H} = L^2(\mathbf{R}^n)$. Then H is self-adjoint and for every $f \in \mathbf{H}$

$$\left(e^{-tH} f\right)(x) = \int_{\Omega_x} f(\omega(t)) \exp\left(\int_0^t V(\omega(s))ds\right) d_x\omega \tag{32}$$

where Ω_x is the space of continuous paths emanating from x and $d_x\omega$ is the associated Wiener measure.

Proof. [From Reed-Simon II page 280.] Since multiplication by V is a bounded self-adjoint operator, we can apply the Kato-Rellich theorem (with $a = 0$!) to conclude that H is self-adjoint, and with the same domain as Δ. So we may apply the Trotter product formula to conclude that

$$(e^{-Ht})f = \lim_{m\to\infty}\left(e^{-\frac{t}{m}\Delta}e^{-\frac{t}{m}V}\right)^m f.$$

This convergence is in L^2, but by passing to a subsequence we may also assume that the convergence is almost everywhere.

Now

$$\left[\left(e^{-\frac{t}{m}\Delta}e^{-\frac{t}{m}V}\right)^m f\right](x)$$

$$= \int_{\mathbf{R}^n}\cdots\int_{\mathbf{R}^n} p\left(x, x_m, \frac{t}{m}\right)\cdots p\left(x, x_m, \frac{t}{m}\right) f(x)_1 \exp\left(-\sum_{j=1}^m \frac{t}{m}V(x_j)\right) dx_1\cdots dx_m.$$

By the very definition of Wiener measure, this last expression is

$$\int_{\Omega_x} \exp\left(\frac{t}{m}\sum_{j=1}^m V\left(\omega\left(\frac{jt}{m}\right)\right)\right) f(\omega(t))d_x\omega.$$

The integrand (with respect to the Wiener measure $d_x\omega$) converges on all continuous paths, that is to say almost everywhere to $\int_{\Omega_x} f(\omega(t))\exp\left(\int_0^t V(\omega(s))ds\right)d_x\omega$

(32)

So to justify (32) we must prove that

the integral of the limit is the limit of the integral. We will do this by the dominated convergence theorem:

$$\int_{\Omega_x} \left| \exp\left(\frac{t}{m} \sum_{j=1}^{m} V\left(\omega\left(\frac{jt}{m} \right) \right) \right) f(\omega(t)) \right| d_x\omega$$

$$\leq e^{t \max |V|} \int_{\Omega_x} |f(\omega(t))| d_x\omega = e^{t \max |V|} \left(e^{-t\Delta} |f| \right)(x) < \infty$$

for almost all x. Hence, by the dominated convergence theorem, (32) holds for almost all x. QED