Math 213a Final Examination, January 2005

INSTRUCTIONS. *For this take-home final examination you are allowed to consult the lecture notes from the course, your homework assignments, and any textbooks and reference material which you have been using for this course prior to the final examination, but you are not allowed to consult other people or go to the library, the internet, or other sources to look up methods and answers to help you with the problems of the final examination. Please put your answers by 5 p.m., January 14, 2005 (Friday) in the mailbox of Professor Yum-Tong Siu in the panel of mailboxes to the right of the main office of the Mathematics Department on the third floor of the Science Center.*

PROBLEM 1. Evaluate

$$\int_0^\infty e^{\cos x} \sin(\sin x) \frac{dx}{x}$$

to verify that the answer is a polynomial in e and π with rational coefficients.

PROBLEM 2. Prove that

$$\frac{e^{az}}{e^z - 1} = \frac{1}{z} + \sum_{n=1}^\infty \frac{2z\cos 2n\pi a - 4n\pi \sin 2n\pi a}{z^2 + 4n^2\pi^2}$$

for $0 < a < 1$.

PROBLEM 3. Prove that

$$\frac{\cosh k - \cos z}{1 - \cos z} = \prod_{n=-\infty}^\infty \left(1 + \left(\frac{k}{2n\pi - z}\right)^2\right)$$

for k, $z \in \mathbb{C}$. (*Hint:* Consider the left-hand side as a function of k.)

PROBLEM 4 For $f(z)$ be a holomorphic function on $\{|z| < 1\}$. For $0 \le r < 1$ define

$$\Phi(r) = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right| d\theta,$$
$$\Psi(r) = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta.$$

(a) Show that $\Psi(r)$ is a nondecreasing function of r and that $\log \Psi(r)$ is a convex functions of $\log r$.

(b) Show that $\Phi(r)$ is a nondecreasing function of r and that $\log \Phi(r)$ is a convex functions of $\log r$.

(c) Let $M(r)$ be the supremum of $|f(z)|$ on $\{|z| = r\}$. Show that $\log M(r)$ is a convex function of $\log r$.

Recall: A function $\phi(x)$ is convex if

$$\phi(x) \leq \frac{x_2 - x}{x_2 - x_1} \phi(x_1) + \frac{x - x_1}{x_2 - x_1} \phi(x_2) \quad \text{for } x_1 < x < x_2$$

or equivalently, when ϕ is twice differentiable, the second derivative of ϕ is nonnegative.

Hint: For the convexity in (a), use the second derivative criterion. For the convexity in (b), to verify the other criterion, choose $g(\theta)$ so that $\Phi(r)$ is the average $F(z)$ of $f\left(ze^{i\theta}\right) g(\theta)$ over $0 \leq \theta \leq 2\pi$ at $z = r$ and apply the maximum principle to $|z^\alpha F(z)|$ for some appropriate real number α.

PROBLEM 5. For a lattice $L = \mathbb{Z}\omega + \mathbb{Z}\omega'$, the Weierstrass $\mathfrak{P}$-function and ζ-function are defined as follows.

$$\mathfrak{P}(w) = \frac{1}{w^2} + \sum_{\ell \in L - 0} \left(\frac{1}{(w-\ell)^2} - \frac{1}{\ell^2} \right),$$

$$\zeta(w) = \frac{1}{w} + \sum_{\ell \in L - 0} \left(\frac{1}{w-\ell} + \frac{1}{\ell} + \frac{w}{\ell^2} \right).$$

(a) Prove by Liouville's theorem that, if x, y, $z \in \mathbb{C}$ with $x + y + z = 0$, then

$$(\zeta(x) + \zeta(y) + \zeta(z))^2 + \zeta'(x) + \zeta'(y) + \zeta'(z) = 0.$$

Hint: $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ when $a + b + c = 0$.

(b) Verify that

$$\begin{vmatrix} 1 & \mathfrak{P}(x) & \mathfrak{P}'(x) \\ 1 & \mathfrak{P}(y) & \mathfrak{P}'(y) \\ 1 & \mathfrak{P}(z) & \mathfrak{P}'(z) \end{vmatrix} = 0$$

for x, y, $z \in \mathbb{C}$ with $x + y + z = 0$.

Hint: Consider complex numbers a and b such that

$$\mathfrak{P}'(w) + a\mathfrak{P}(w) + b = 0 \quad \text{for } w = x \text{ and } w = y$$

and consider the pole-set of $\mathfrak{P}'(w) + a\mathfrak{P}(w) + b$.

PROBLEM 6. The theta functions of Jacobi $\vartheta_\nu(w)$ for $\nu = 1, 2, 3, 4$ are given by

$$\vartheta_1(w) = 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2} \sin(2n+1)w,$$

$$\vartheta_2(w) = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} \cos(2n+1)w,$$

$$\vartheta_3(w) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nw,$$

$$\vartheta_4(w) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos 2nw,$$

where $q = e^{i\pi\tau}$ for some complex number τ with positive imaginary part.

Verify that

$$\vartheta_1(2w)\vartheta_2(0)\vartheta_3(0)\vartheta_4(0) = 2\vartheta_1(w)\vartheta_2(w)\vartheta_3(w)\vartheta_4(w)$$

for $w \in \mathbb{C}$.

PROBLEM 7. For a complex number τ with positive imaginary part, introduce the lattice $L = \mathbb{Z} + \mathbb{Z}\tau$ and the two complex numbers $g_2(\tau)$ and $g_3(\tau)$ defined by

$$g_2(\tau) = 60 \sum_{\ell \in L-0} \frac{1}{\ell^4},$$

$$g_3(\tau) = 140 \sum_{\ell \in L-0} \frac{1}{\ell^6}.$$

Consider the elliptic modular function

$$J(\tau) = \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}.$$

Let $\mathbb{H}$ be the open upper half-plane $\{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$. By using the topological covering map

$$J : \mathbb{H} - J^{-1}(\{0,1\}) \to \mathbb{C} - \{0,1\},$$

prove the following theorem of Schottky.

There exists a positive function $A(a_0, r)$ depending on $a_0 \in \mathbb{C}$ and $0 < r < 1$ such that for every holomorphic function $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ on the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$ which does not assume the values 0 and 1, the inequality $|f(z)| \leq A(a_0, r)$ holds on $|z| \leq r$.