Math 213a Homework December 10, 2004

Notations. $\mathbb{C}$ is the set of all complex numbers and $\mathbb{Z}$ is the set of all integers.

Problem 1. The complex projective space $\mathbb{P}_n$ is defined as the quotient of $\mathbb{C}^{n+1} - 0$ by the action of $\mathbb{C} - 0$ which sends $(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} - 0$ to $\lambda(z_0, \cdots, z_n)$ for $\lambda \in \mathbb{C} - 0$. Let U_ν be the open subset of $\mathbb{P}_n$ consisting of the images of all points $(z_0, \cdots, z_n)$ with z_ν nonzero. Denote by $[z_0 : \cdots : z_n]$ the point of $\mathbb{P}_n$ which is the image of the point $(z_0, \cdots, z_n)$ of $\mathbb{C}^{n+1} - 0$. The space $\mathbb{P}_n$ is made into a complex manifold by endowing U_ν with the global holomorphic coordinates $\frac{z_j}{z_\nu}$ for $0 \leq j \leq n$ and $j \neq \nu$. Let $F(z_0, \cdots, z_n)$ be a homogeneous polynomial of $n+1$ variables $z_0, \cdots, z_n$. Let V be the subset of $\mathbb{P}_n$ consisting of all points $[z_0 : \cdots : z_n]$ with $F(z_0, \cdots, z_n) = 0$. Show that the following two statements are equivalent:

(i) For $0 \leq \nu \leq n$, the set $V \cap U_\nu$ is a submanifold of U_ν (*i.e.*, the gradient of

$$F\left(\frac{z_0}{z_\nu}, \cdots, \frac{z_{\nu-1}}{z_\nu}, 1, \frac{z_{\nu+1}}{z_\nu}, \cdots, \frac{z_n}{z_\nu}\right)$$

with respect to the coordinates $\frac{z_j}{z_\nu}$ $(0 \leq j \leq n, j \neq \nu)$ is nowhere zero on $V \cap U_\nu$.

(ii) On $\mathbb{C}^{n+1}$ the gradient of $F(z_0, \cdots, z_n)$ with respect to $(z_0, \cdots, z_n)$ is nowhere zero on the set $F(z_0, \cdots, z_n) = 0$ unless $(z_0, \cdots, z_n) = 0$.

Problem 2. Let $F_d(x)$ be a polynomial of degree $d \geq 3$ with complex coefficients whose roots are all distinct. The Riemann surface M_{F_d} for $\sqrt{F_d(x)}$ is constructed by taking disjoint slits in the Riemann sphere joining either two roots or one root and ∞ so that every root belongs to precisely one slit and glueing together two copies of the slitted Riemann sphere in such a way that the sides of corresponding slits are glued in a criss-cross manner. Let g be the largest integer such that $2g \leq d - 1$. Prove that the differential form $\omega_\nu = \frac{x^\nu dx}{\sqrt{F_d(x)}}$ is holomorphic on the Riemann surface M_{F_d} for $0 \leq \nu < g$. For each $0 \leq \nu < g$ find the total number of zeroes of ω_ν on M_{F_d} with multiplicity counted. When d is even, show that for any polynomial F_d there exists some polynomial F_{d-1} so that some Möbius transformation in x maps M_{F_d} to $M_{F_{d-1}}$ biholomorphically with the $\mathbb{C}$-vector space spanned by $\frac{x^\nu dx}{\sqrt{F_d(x)}}$ $(0 \leq \nu < g)$ mapped to the $\mathbb{C}$-vector space spanned by $\frac{x^\nu dx}{\sqrt{F_{d-1}(x)}}$ $(0 \leq \nu < g)$.

Problem 3. Let M and N be compact Riemann surfaces of genus m and n respectively. (The *genus* of a compact oriented surface means one-half of the first Betti number or equivalently the number of "holes" in the surface.) Let $f : M \to N$ be a nonconstant holomorphic map. Let Z be the set of points where f is not locally biholomorphic. Let k be the *total branching order* of the map $f : M \to N$, which is defined as the sum of the vanishing order of the differential df of f at all the points of Z. Let λ be *the number of sheets* of the map $f : M \to N$, which is defined as the number of points in $f^{-1}(w)$ for $w \in N - f(Z)$. By using triangulations of M and N in which every point of Z and $f(Z)$ is a vertex, show by comparing the Euler numbers of M and N that $m = \lambda n - \lambda + 1 + \frac{k}{2}$.

Problem 4. Let $F(x, y)$ $(x, y \in \mathbb{C})$ be a polynomial of degree d defining a *nonsingular* complex curve C in the two-dimensional complex projective space $\mathbb{P}_2$. Show that the genus of C is $\frac{1}{2}(d-1)(d-2)$. Explain why the Riemann surface constructed in Problem 2 for the square root of a polynomial of degree 4 has genus 1 instead of 3. (Hint: Take a suitable point P in $\mathbb{P}_2$ and a suitable line L inside $\mathbb{P}_2$. Apply Problem 3 to the map which projects C onto L by using P as the light source and count the number of branching points and the number of sheets. For the explanation, consider whether $y^2 = F_4(x)$ defines a nonsingular curve in $\mathbb{P}_2$.)

Problem 5. Let $F(x, y)$ and C be as in Problem 4. Let g be the genus of C. By using $F_x dx + F_y dy \equiv 0$ on $F(x, y) = 0$ (where F_x and F_y are respectively the partial derivatives of F with respect to x and y), show that there are g $\mathbb{C}$-linearly independent holomorphic 1-forms on C of the form $\frac{P(x,y)dx}{F_y(x,y)}$, where $P(x, y)$ is a polynomial in x and y.

Problem 6. The special linear group

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

acts on the upper half-plane H by Möbius transformations. Let M be the one-point compactification of the quotient $H/SL(2, \mathbb{Z})$. Let z be the coordinate of H and let ζ_1, ζ_2, ζ_3 be defined by $z = (\zeta_1 - i)^2$, $z = (\zeta_2 - e^{\pi i/3})^3$, $z = e^{2\pi i \zeta_3}$. By using ζ_1 (respectively ζ_2, ζ_3) to define the local coordinate of M at the image of i (respectively the image of $e^{\pi i/3}$ and the point at infinity), show that M is a compact Riemann surface. Find the genus of M. (Hint: consider the elliptic modular function J.)