Math 213a Homework December 17, 2004

Problem 1 (Embedding of Riemann Surfaces in $\mathbb{P}_N$). Let M be a compact Riemann surface whose universal cover is the open unit disk Δ so that M is represented as the quotient of Δ by a subgroup Γ of Möbius transformations. Let $\Omega \subset \Delta$ be a fundamental domain for Γ so that Ω is a bounded domain together with some of its boundary points and Δ is the disjoint union of $\gamma\,(\Omega)$ for $\gamma \in \Gamma$. For a bounded holomorphic function φ on Δ and every integer $k \geq 2$ let

$$P_{k,\varphi,\Gamma}(z) = \sum_{\gamma \in \Gamma} \varphi(\gamma(z))\,(\gamma'(z))^k$$

denote the Poincaré series.

(a) Let r_0 be the supremum of $|z|$ for $z \in \Omega$. Let $r_0 < r_1 < 1$. Verify that there exists a finite subset Γ_{r_1} of Γ such that $\gamma(\Omega)$ is contained in $\{z \in \Delta \,|\, |z| \geq r_1\}$ for $\gamma \in \Gamma - \Gamma_{r_1}$. (In other words, $\gamma(\Omega)$ intersects $\{z \in \Delta \,|\, |z| < r_1\}$ only for a finite number of elements γ of Γ.)

(b) Verify that, for any Möbius transformation T mapping Δ to Δ, one has
$$\frac{|dw|}{1 - |w|^2} = \frac{|dz|}{1 - |z|^2}$$
for $w = T(z)$. Show that, as a consequence,
$$|\gamma'(z)| \leq \frac{1 - r_1^2}{1 - r_0^2}$$
for $z \in \Omega$ and $\gamma \in \Gamma - \Gamma_{r_1}$. (*Hint:* T must be of the form
$$e^{i\alpha}\frac{z - a}{1 - \bar{a}z}$$
for some $a \in \Delta$ and $\alpha \in \mathbb{R}$.)

(c) For two points P and Q of Ω whose images in M are distinct, let $\varphi_{P,Q}$ be a polynomial whose value at P is 1 and whose value at $\gamma(P)$ is 0 for $\gamma \in \Gamma_{r_1} - \mathrm{id}_\Delta$ and whose value at $\gamma(Q)$ is 0 for $\gamma \in \Gamma_{r_1}$, where id_Δ is the identity map of Δ. Show that, given any $\varepsilon > 0$ there exists an integer $k_{\varepsilon,P,Q} \geq 2$ such that, for $k \geq k_{\varepsilon,P,Q}$,
$$\left|1 - P_{k,\varphi_{P,Q},\Gamma}(P)\right| < \varepsilon$$

and
$$\left|P_{k,\varphi_{P,Q},\Gamma}(Q)\right| < \varepsilon.$$

(*Hint:* The value of
$$\sum_{\gamma \in \Gamma_{r_1}} \varphi(\gamma(z))\,(\gamma'(z))^k$$

at P is equal to 1 and its value at Q is equal to 0 for all choices of $k \geq 2$. By Part (b) the value of

$$\sum_{\gamma \in \Gamma - \Gamma_{r_1}} \varphi(\gamma(z))\,(\gamma'(z))^k$$

approaches 0 at both P and Q as $k \to \infty$.)

(d) For point P of Ω let φ_Q be a polynomial whose value at P is 0 and whose derivative at P is 1 such that both its value and its derivative vanish at $\gamma(P)$ for $\gamma \in \Gamma_{r_1} - \mathrm{id}_\Delta$. Show that, given any $\varepsilon > 0$ there exists an integer $k_{\varepsilon,P} \geq 2$ such that, for $k \geq k_{\varepsilon,P}$,

$$\left| 1 - \left(\frac{d}{dz} P_{k,\varphi_P,\Gamma}(z) \right)_{z=P} \right| < \varepsilon$$

and
$$\left|P_{k,\varphi_P,\Gamma}(P)\right| < \varepsilon.$$

(e) Show that, by using a finite number of finite products whose factors are Poincaré series of the form $P_{k_j,\varphi_{P_j},\Gamma}$ or $P_{k_\ell,\varphi_{P_\ell,Q_\ell},\Gamma}$, a holomorphic map from M to some $\mathbb{P}_N$ can be defined which is an embedding. (The factors of some products used in the definition of the map may all be of the form $P_{k_j,\varphi_{P_j},\Gamma}$. The factors of some products used in the definition of the map may all be of the form $P_{k_\ell,\varphi_{P_\ell,Q_\ell},\Gamma}$. Some other products used in the definition of the map may contain some $P_{k_j,\varphi_{P_j},\Gamma}$ as factors and at the same time some $P_{k_\ell,\varphi_{P_\ell,Q_\ell},\Gamma}$ as factors.)

Problem 2 (Number of Zeroes of a Holomorphic 1-Form). Let M be a compact Riemann surface of genus $g \geq 1$ so that its Euler number is $2 - 2g$. Suppose that M is holomorphically embedded in some $\mathbb{P}_N$. Choose some linear $\mathbb{P}_{N-2}$ in $\mathbb{P}_N$ which is disjoint from M and call it L. Choose some linear $\mathbb{P}_1$ disjoint from L and call it S. Let $\pi : M \to S$ be defined as follows.

For $P \in M$, $\pi(P)$ is the intersection point of S and the linear $\mathbb{P}_{N-1}$ which contains x and L. Let Z be the set of points of M where π is not locally biholomorphic. Choose a meromorphic function f on S with only a single pole whose order is 1 and which is located at a point of $S - \pi(Z)$. Consider a triangulation of M and a triangulation of S so that their triangles correspond under π and every point of Z is a vertex of the triangulation of M. By using the two triangulations to compare the Euler numbers of M and S and by considering the zeroes and the poles of the meromorphic 1-form $\pi^* (df)$ on M, show that any holomorphic 1-form on M has precisely $2g - 2$ zeroes when multiplicities are counted.