Math 213a Homework December 3, 2004

Problem 1 (Riemann Mapping Theorem and Bergman Kernel). Let Ω be a simply connected bounded domain in $\mathbf{C}$. Let $f_\nu(z)$ $(1 \leq \nu < \infty)$ be an orthonormal basis of the Hilbert space of all square integrable holomorphic functions on Ω.

(a) Show that $\sum_{\nu=1}^{\infty} |f_\nu(z)|^2$ converges uniformly on compact subsets of Ω. (Hint: for any relatively compact disk D in Ω, use the fact that for any holomorphic function g on Ω the value $|g|^2$ at the center c of D is no more than the average of $|g|^2$ on D and apply this to the case $g = \sum_{\nu=1}^{n} a_\nu f_\nu(z)$ with $a_\nu = \overline{f_\nu(c)}$.)

(*b*) Define $K(z, \bar{w}) = \sum_{\nu=1}^{\infty} f_\nu(z)\overline{f_\nu(w)}$ and show that the series converges on uniformly on compact subsets of $\Omega \times \Omega$ and $K(z, \bar{w})$ is holomorphic in z and anti-holomorphic in w. The function $K(z, \bar{w})$ is known as the Bergman kernel function. Show that for any point w_0 in Ω the map

$$z \mapsto \left(\frac{\partial}{\partial \bar{w}} K(z, \bar{w}) \right)_{w=w_0} K(z, \bar{w})^{-1}$$

maps Ω biholomorphically onto an open disk in $\mathbf{C}$. (Hint: use the uniformization theorem (or the Riemann mapping theorem which says that any simply connected proper subdomain of $\mathbb{C}$ is biholomorphic to the open unit 1-disk) and observe how the Bergman kernel function transforms under a biholomorphic map and write down the Bergman kernel function for the open unit disk from definition.)

Problem 2. Let $f(z)$ be a univalent (*i.e.,* holomorphic injective) function on the open unit disk Δ in $\mathbf{C}$. Show that

$$\left| \frac{\partial}{\partial r} \arg f'(z) \right| \leq \frac{4}{1 - r^2},$$

where $r = |z|$ and $\arg f'(z)$ means the argument of $f'(z)$. (Hint: use the analog of the derivation of

$$\frac{\partial}{\partial r} \log |f'(z)| \leq \frac{2(2 + r)}{1 - r^2}$$

in the Koebe distortion theorem in the lectures, but use the imaginary part instead of the real part.)

Problem 3 (Characterization of Poisson Kernel). Suppose Ω be a bounded domain in $\mathbf{C}$ with smooth boundary. Let $P(z, \zeta)$ be a nonnegative continuous function for $z \in \Omega$ and $\zeta \in \partial\Omega$ so that its second-order partial derivatives in z are continuous on $\Omega \times \partial\Omega$. Suppose the following conditions are satisfied.
(a) For $\zeta \in \partial\Omega$ the function $P(z, \zeta)$ as a function of z is harmonic on Ω.
(b) For $z \in \Omega$ the integral of $P(z, \zeta)$ over $\zeta \in \partial\Omega$ is 1.
(c) For $\zeta \in \partial\Omega$ and $r > 0$, $P(z, \zeta)$ approaches 0 as z approaches $\partial\Omega - \Delta_r(\zeta)$ uniformly in ζ, where $\Delta_r(\zeta)$ is the disk of radius r centered at ζ (*i.e.,* for any $\epsilon > 0$ there exists $\delta > 0$ independent of $\zeta \in \partial\Omega$ such that $P(z, \zeta) < \epsilon$ if $|z - \zeta| > r$ and $\operatorname{dist}(z, \partial\Omega) < \delta$).
Show that $P(z, \zeta)$ is the Poisson kernel for Ω in the sense that when $u(\zeta)$ is a continuous function on $\partial\Omega$ the function $v(z) = \int_{\partial\Omega} P(z, \zeta)u(\zeta)$ is harmonic in z and its boundary value is equal to $u(\zeta)$. Verify that $\frac{1}{2\pi}$ Re $\frac{\zeta+z}{\zeta-z}$ is the Poisson kernel for the open unit disk in $\mathbf{C}$.

Problem 4 (Generalization of Poisson Kernel). Prove the followiong formula of Poisson-Jensen. Let f be a meromorphic function on an open neighborhoood of the closed disk $|z| \leq R$ and let its zeroes be $a_1, \cdots, a_m$ and its poles be $b_1, \cdots, b_n$ on $|z| \leq R$. Then

$$\log|f(re^{i\theta})| = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{R^2 - r^2}{|Re^{i\varphi} - re^{i\theta}|^2} \log|f(Re^{i\varphi})| d\varphi$$
$$- \sum_{\mu=1}^{m} \log\left|\frac{R^2 - \bar{a}_\mu^{-} re^{i\theta}}{R(e^{i\theta} - a_\mu)}\right| + \sum_{\nu=1}^{n} \log\left|\frac{R^2 - \bar{b}_\nu^{-} re^{i\theta}}{R(e^{i\theta} - b_\nu)}\right|.$$

(Hint: consider first the case where f is nowhere zero holomorphic and then take out the zeros and poles by removing factors constructed from Moebius transformations.)

Problem 5 (Dirichlet Problem for the Annulus). Let D be the annulus $r < |z| < s$, where $0 < r < s < \infty$. Let f be a continuous function on the circle $|z| = r$ and g be a continuous function on the circle $|z| = s$. Determine the coefficients a_k and b (with a_0 real) in

$$h = b\ \log|z| + \text{Re} \sum_{k=-\infty}^{\infty} a_k z^k$$

in terms of the Fourier coefficients of f and g so that h is harmonic on D and its boundary values are given by f and g.

Problem 6. For $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$ let $L = \mathbb{Z} + \mathbb{Z}\tau$ and $g_2 = 60 \sum_{\ell \in L-0} \frac{1}{\ell^4}$ and $g_3 = 140 \sum_{\ell \in L-0} \frac{1}{\ell^6}$. Define the elliptic modular function $J(\tau)$ as $\frac{g_2^3}{g_2^3 - 27 g_3^2}$. Prove the following Big Picard Theorem by using the ellitpic modular function $J(\tau)$. Any entire function f on $\mathbf{C}$ whose image misses two distinct complex numbers must be constant. (Hint: reduce it to the special case where the entire function misses 0 and 1 in its image and then consider a branch of the function $J^{-1} \circ f$ whose image is a bounded subset of $\mathbf{C}$. Note that J is locally biholomorphic outside of $J^{-1}(\{0, 1\})$.)

Problem 7. The Riemann mapping theorem states any domain D in $\mathbf{C}$ whose complement contains at least two distinct points is biholomorphic to the open unit disk $\Delta = \{|z| < 1\}$ under some holomorphic map $f : D \to \Delta$. Let P be a point in D. Prove that such a map f is unique if normalized to satisfy $f(P) = 0$ and $f'(P) =$ real. (Hint: consider $g \circ f^{-1}$ if g is another such map.)

Problem 8. Let k be a nonzero complex number whose argument is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that k^2 does not belong to the line segment $[1, \infty)$ on the real axis. The function $x = \operatorname{sn} w = \operatorname{sn}\,(w, k)$ is defined by the differential equation

$$\frac{d}{dw} \operatorname{sn} w = \sqrt{(1 - \operatorname{sn}^2 w)(1 - k^2 \operatorname{sn}^2 w)}$$

with the initial value of $\operatorname{sn} w = 0$ at $w = 0$ and the square root on the right-hand side chosen to be 1 at $\operatorname{sn} w = 0$. Let

$$K = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)\,(1 - k^2 x^2)}}\ ,$$

where the integration is along the straight line from 0 to 1 and the value of the square root in the denominator assumes the value 1 at the point $x = 0$ after the two slits of the line segment joining 1 to $\frac{1}{k}$ and the line segment joining -1 to $\frac{-1}{k}$ are made in the x-plane. Let

$$K' = -\sqrt{-1} \int_1^{\frac{1}{k}} \frac{dx}{\sqrt{(1 - x^2)\,(1 - k^2 x^2)}}\ ,$$

where the value of the square root is as in the definition of K and the integration is along the line segment which is part of the limit of the loop circling the slit from 1 to $\frac{1}{k}$ in the counter-clockwise sense. Show that $w = \operatorname{sn} z$ maps the interior of the rectangle with vertices K, $K + i\,K'$, $-K + i\,K'$, $-K$ biholomorphically onto the upper half-plane. (Hint: consider the correspondence between z and w when both trace the boundary of the two domains.)

Problem 9. Let

$$z = \int_{\zeta=0}^{w} \frac{d\zeta}{\sqrt{\zeta(1-\zeta^2)}}.$$

Show that, with an appropriate interpretation of the above integral, $z \mapsto w$ maps the interior of a square of side

$$\int_{\theta=0}^{\pi/2} (\operatorname{cosec}\theta)^{1/2}\, d\theta$$

to the upper half-plane.