Math 213a Homework November 12, 2004

Notations. Let k be a nonzero complex number whose argument is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that k^2 does not belong to the line segment $[1, \infty)$ on the real axis. Let k' be $\sqrt{1-k^2}$, where the function $\sqrt{1-z}$ is defined with the slit $[1, \infty)$ on the real axis removed and its value is defined as 1 at $z = 0$.

The function $x = \operatorname{sn} w = \operatorname{sn}\,(w, k)$ is defined by the differential equation

$$\frac{d}{dw} \operatorname{sn} w = \sqrt{(1 - \operatorname{sn}^2 w)(1 - k^2 \operatorname{sn}^2 w)}$$

with the initial value of $\operatorname{sn} w = 0$ at $w = 0$ and the square root on the right-hand side chosen to be 1 at $\operatorname{sn} w = 0$.

The function $\operatorname{cn} w =$ is defined by $\operatorname{cn} w = \sqrt{1 - \operatorname{sn}^2 w}$ with the initial value of $\operatorname{cn} w = 1$ at $w = 0$.

The function $\operatorname{dn} w =$ is defined by $\operatorname{dn} w = \sqrt{1 - k^2 \operatorname{sn}^2 w}$ with the initial value of $\operatorname{dn} w = 1$ at $w = 0$.

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}\,,$$

where the integration is along the straight line from 0 to 1 and the value of the square root in the denominator assumes the value 1 at the point $x = 0$ after the two slits of the line segment joining 1 to $\frac{1}{k}$ and the line segment joining -1 to $\frac{-1}{k}$ are made in the x-plane.

$$K' = -\sqrt{-1} \int_1^{\frac{1}{k}} \frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}\,,$$

where the value of the square root is as in the definition of K and the integration is along the line segment which is part of the limit of the loop circling the slit from 1 to $\frac{1}{k}$ in the counter-clockwise sense.

Problem 1. Assume that the following addition formula for $\operatorname{sn} w$ is known:

$$\operatorname{sn}\,(w_1 + w_2) = \frac{\operatorname{sn} w_1 \operatorname{cn} w_2 \operatorname{dn} w_2 + \operatorname{sn} w_2 \operatorname{cn} w_1 \operatorname{dn} w_1}{1 - k^2 \operatorname{sn}^2 w_1 \operatorname{sn}^2 w_2}.$$

Verify the following addition formulae for $\mathrm{cn}\, w$ and $\mathrm{dn}\, w$ by using $\mathrm{cn}^2 w = 1 - \mathrm{sn}^2 w$ and $\mathrm{dn}^2 w = 1 - k^2 \mathrm{sn}^2 w$ and continuity and special values to determine the signs of square roots.

$$\begin{aligned}
\mathrm{cn}\,(w_1 + w_2) &= \frac{\mathrm{cn}\, w_1\, \mathrm{cn}\, w_2 - \mathrm{sn}\, w_1\, \mathrm{sn}\, w_2\, \mathrm{dn}\, w_1\, \mathrm{dn}\, w_1}{1 - k^2 \mathrm{sn}^2 w_1\, \mathrm{sn}^2 w_2}, \\
\mathrm{dn}\,(w_1 + w_2) &= \frac{\mathrm{dn}\, w_1\, \mathrm{dn}\, w_2 - k^2 \mathrm{sn}\, w_1\, \mathrm{sn}\, w_2\, \mathrm{dn}\, w_1\, \mathrm{dn}\, w_1}{1 - k^2 \mathrm{sn}^2 w_1\, \mathrm{sn}^2 w_2}.
\end{aligned}$$

Problem 2. (a) Use the definitions of K, $\sqrt{-1}\, K'$, k', $\mathrm{cn}\, w$, and $\mathrm{dn}\, w$ to verify that

$$\begin{gathered}
\mathrm{sn}(K) = 1, \quad \mathrm{cn}(K) = 0, \quad \mathrm{dn}(K) = k', \\
\mathrm{sn}(K + \sqrt{-1}\, K') = \frac{1}{k}, \quad \mathrm{cn}(K + \sqrt{-1}\, K') = -\sqrt{-1}\, \frac{k'}{k}, \quad \mathrm{dn}(K + \sqrt{-1}\, K') = 0.
\end{gathered}$$

(b) Use the addition formulas to verify that

$$\mathrm{sn}\,(w + K) = \frac{\mathrm{cn}\, w}{\mathrm{dn}\, w}, \quad \mathrm{cn}\,(w + K) = -k' \frac{\mathrm{sn}\, w}{\mathrm{dn}\, w}, \quad \mathrm{dn}\,(w + K) = \frac{k'}{\mathrm{dn}\, w},$$

and

$$\begin{gathered}
\mathrm{sn}\left(w + \sqrt{-1}\, K'\right) = \frac{1}{k\, \mathrm{cn}\, w}, \quad \mathrm{cn}\left(w + \sqrt{-1}\, K'\right) = -\frac{\sqrt{-1}}{k} \frac{\mathrm{dn}\, w}{\mathrm{sn}\, w}, \\
\mathrm{dn}\left(w + \sqrt{-1}\, K'\right) = -\sqrt{-1}\, \frac{\mathrm{cn}\, w}{\mathrm{sn}\, w}.
\end{gathered}$$

(c) Verify that the periods of $\mathrm{sn}\, w$ are $4K$ and $2\sqrt{-1}\, K'$, the periods of $\mathrm{cn}\, w$ are $4K$ and $2K + 2\sqrt{-1}\, K'$, the periods of $\mathrm{dn}\, w$ are $2K$ and $4\sqrt{-1}\, K'$.

Problem 3. Derive the following identities:

$$\begin{aligned}
\mathrm{sn}\left((1+k')\sqrt{-1}\, w, \frac{2\sqrt{k'}}{1+k'}\right) &= \frac{\sqrt{-1}\,(1+k')\mathrm{sn}(w,k)\mathrm{cn}(w,k)}{1-(1+k')\mathrm{sn}^2(w,k)} \\
\mathrm{cn}\left((1+k')\sqrt{-1}\, w, \frac{2\sqrt{k'}}{1+k'}\right) &= \frac{\mathrm{dn}(w,k)}{1-(1+k')\mathrm{sn}^2(w,k)} \\
\mathrm{dn}\left((1+k')\sqrt{-1}\, w, \frac{2\sqrt{k'}}{1+k'}\right) &= \frac{1-(1-k')\mathrm{sn}^2(w,k)}{1-(1+k')\mathrm{sn}^2(w,k)}.
\end{aligned}$$

(Hint: for the first identity, let $z = \operatorname{sn}(w, k)$ and

$$t = \operatorname{sn}\left((1+k')\sqrt{-1}\,w, \frac{2\sqrt{k'}}{1+k'}\right)$$

and show that the substitution of

$$t = \frac{\sqrt{-1}\,(1+k')z\sqrt{1-z^2}}{1-(1+k')z^2}$$

(which is a restatement of the identity) transforms

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \quad \text{to} \quad \frac{1}{1+k'}\frac{dt}{\sqrt{(1-t^2)(1-\ell^2z^2)}},$$

where $\ell = \frac{2\sqrt{k'}}{1+k'}$.)

Problem 4. Show that

$$\int \operatorname{sn} w\, dw = \frac{1}{2k}\log\frac{1-k\,\frac{\operatorname{cn} w}{\operatorname{dn} w}}{1+k\,\frac{\operatorname{cn} w}{\operatorname{dn} w}}$$

$$\int \operatorname{cn} w\, dw = \frac{1}{k}\arctan\left(k\,\frac{\operatorname{sn} w}{\operatorname{dn} w}\right)$$

$$\int \frac{\operatorname{sn} w}{\operatorname{cn} w}\, dw = \frac{1}{2k'}\log\frac{\operatorname{dn} w + k'}{\operatorname{dn} w - k'}$$

$$\int \frac{\operatorname{dn} w}{\operatorname{sn} w}\, dw = \frac{1}{2}\log\frac{1-\operatorname{cn} w}{1+\operatorname{cn} w}$$

$$\int \frac{\operatorname{dn} w}{\operatorname{cn} w}\, dw = \frac{1}{2}\log\frac{1+\operatorname{sn} w}{1-\operatorname{sn} w}.$$