Notations. τ is a complex number with $\operatorname{Im}\tau > 0$ and $q = e^{i\pi\tau}$.

$$\vartheta_1(w,\tau) = -i\,e^{iw+\frac{1}{4}\pi i\tau}\vartheta_4\left(w + \frac{1}{2}\pi\tau\right) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n e^{\frac{1}{4}(2n+1)^2 i\pi\tau}e^{(2n+1)iw}$$

$$= 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)w,$$

$$\vartheta_2(w,\tau) = \vartheta_1\left(w + \frac{\pi}{2}\right) = \sum_{n=-\infty}^{\infty} e^{\frac{1}{4}(2n+1)^2 i\pi\tau}e^{(2n+1)iw} = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}\cos(2n+1)w,$$

$$\vartheta_3(w,\tau) = \vartheta_4\left(w + \frac{\pi}{2}\right) = \sum_{n=-\infty}^{\infty} e^{n^2 i\pi\tau}e^{2niw} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nw.$$

$$\vartheta_4(w,\tau) = \sum_{n=-\infty}^{\infty}(-1)^n e^{n^2 i\pi\tau}e^{2niw} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nw.$$

Problem 1. (a) Verify the following "Jacobi's imaginary transformation of the ϑ_3 function"

$$\sqrt{\frac{\tau}{i}}\,\vartheta_3(w,\tau) = \exp\left(\frac{-iw^2}{\pi\tau}\right)\vartheta_3\left(\frac{w}{\tau}, -\frac{1}{\tau}\right),$$

which was used in the proof of the "quadratic reciprocity" presented in class.

(b) By using translations by half periods, derive Jacobi's imaginary transformation for the other three ϑ functions.

$$\sqrt{\frac{\tau}{i}}\,\vartheta_1(w,\tau) = -\exp\left(\frac{-iw^2}{\pi\tau}\right)\vartheta_1\left(\frac{w}{\tau}, -\frac{1}{\tau}\right),$$

$$\sqrt{\frac{\tau}{i}}\,\vartheta_2(w,\tau) = \exp\left(\frac{-iw^2}{\pi\tau}\right)\vartheta_4\left(\frac{w}{\tau}, -\frac{1}{\tau}\right),$$

$$\sqrt{\frac{\tau}{i}}\,\vartheta_4(w,\tau) = \exp\left(\frac{-iw^2}{\pi\tau}\right)\vartheta_2\left(\frac{w}{\tau}, -\frac{1}{\tau}\right).$$

(c) Verify the following Jacobi's imaginary transformation of the ϑ functions for the element $\tau \mapsto \frac{\tau}{1-\tau}$ of the modular group.

$$\vartheta_1\left(w, \frac{\tau}{1-\tau}\right) = i^{\frac{1}{2}}(1-\tau)^{\frac{1}{2}}e^{\frac{iw^2(\tau-1)}{\pi}}\vartheta_1\left((1-\tau)w, \tau\right),$$

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$$\vartheta_2\left(w, \frac{\tau}{1-\tau}\right) = (1-\tau)^{\frac{1}{2}} e^{\frac{iw^2(\tau-1)}{\pi}} \vartheta_3\left((1-\tau)w, \tau\right),$$

$$\vartheta_3\left(w, \frac{\tau}{1-\tau}\right) = (1-\tau)^{\frac{1}{2}} e^{\frac{iw^2(\tau-1)}{\pi}} \vartheta_2\left((1-\tau)w, \tau\right),$$

$$\vartheta_4\left(w, \frac{\tau}{1-\tau}\right) = i^{\frac{1}{2}} (1-\tau)^{\frac{1}{2}} e^{\frac{iw^2(\tau-1)}{\pi}} \vartheta_4\left((1-\tau)w, \tau\right),$$

Problem 2 (Application of Jacobi's elliptic functions to nonlinear waves). To model the 1-dimensional propagation of waves, we use springs connecting adjacent particles, of mass m each, located at the coordinates x_n $(n \in \mathbb{Z})$ of $\mathbb{R}$. Instead of Hooke's law (which describes the force of a spring on its ends as proportional to the change of length of the spring), we assume that there are two positive constants a and b so that the force of the spring between the n-th particle and the $(n+1)$-st particle on its two ends is equal to $a \left| e^{-bx_{n+1}} - e^{-bx_n} \right|$ in magnitude. The sum of the two forces on the n-th particle, one from the left and the other from the right, is

$$-a\left(e^{-bx_{n+1}} - e^{-bx_n}\right) + a\left(e^{-bx_n} - e^{-bx_{n-1}}\right) = a\left(2e^{-bx_n} - e^{-bx_{n+1}} - e^{-bx_{n-1}}\right).$$

The system of differential equations for this model of the 1-dimensional non-linear wave is

$$(*) \qquad m\frac{d^2 x_n}{dt^2} = a\left(2e^{-bx_n} - e^{-bx_{n+1}} - e^{-bx_{n-1}}\right)$$

according to the second law of Newton. Solutions in closed form for this system can be obtained with the use of Jacobi's elliptic functions in the following steps. Do these steps.

(a) Use the addition formula

$$\operatorname{sn}(w_1 + w_2) = \frac{\operatorname{sn} w_1 \operatorname{cn} w_2 \operatorname{dn} w_2 + \operatorname{sn} w_2 \operatorname{cn} w_1 \operatorname{dn} w_1}{1 - k^2 \operatorname{sn}^2 w_1 \operatorname{sn}^2 w_2}$$

(where k is the modulus of the Jacobi elliptic functions sn, cn, and dn) to verify that

$$\operatorname{sn}^2(w_1 + w_2) - \operatorname{sn}^2(w_1 - w_2) = 2\frac{d}{dw_2}\frac{\operatorname{sn} w_1 \operatorname{cn} w_1 \operatorname{dn} w_1 \operatorname{sn}^2 w_2}{1 - k^2 \operatorname{sn}^2 w_1 \operatorname{sn}^2 w_2}.$$

(b) Let

$$F(w) = \int_0^w \operatorname{dn}^2\zeta \, d\zeta.$$

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Use the algebraic relations

$$\operatorname{cn}^2 w = 1 - \operatorname{sn}^2 w,$$
$$\operatorname{dn}^2 w = 1 - k^2 \operatorname{sn}^2 w$$

and the differential relations

$$\frac{d}{dw} \operatorname{sn} w = \operatorname{cn} w \operatorname{dn} w,$$
$$\frac{d}{cw} \operatorname{sn} w = -\operatorname{sn} w \operatorname{dn} w,$$
$$\frac{d}{dw} \operatorname{dn} w = -k^2 \operatorname{sn} w \operatorname{cn} w$$

to integrate the result in Part (a) with respect to w_2 to verify that

$$F(w_1 + w_2) + F(w_1 - w_2) - 2F(w_1) = \frac{\frac{d^2}{dw_1^2} F(w_1)}{\frac{1}{\operatorname{sn}^2 w_2} - 1 + \frac{d}{dw_1} F(w_1)}.$$

(c) By differentiating the result in Part (b) to verify that

$$\operatorname{dn}^2 (w_1 + w_2) + \operatorname{dn}^2 (w_1 - w_2) - 2 \operatorname{dn}^2 w_1 = 2 \frac{d^2}{dw_1} \log \left(\frac{1}{\operatorname{sn}^2 w_2} - 1 + \operatorname{dn}^2 w_1 \right).$$

(d) Let

$$E = \int_0^K \operatorname{dn}^2 w \, dw,$$

where

$$K = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

with an appropriate choice of path of integration. Use the result in Part (c) to verify that a solution of the system of differential equations $(*)$ is given by

$$e^{-bx_n} - 1 = \frac{m(2K\nu)^2}{ab} \left(\operatorname{dn}^2 \left(2 \left(\nu t - \frac{n}{\lambda} \right) K \right) - \frac{E}{K} \right),$$

where the "frequency" ν and the "wave length" λ satisfy the relation

$$(2K\nu)^2 = \frac{ab}{m \left(\frac{1}{\operatorname{sn}^2 \left(\frac{2K}{\lambda} \right)} - 1 + \frac{E}{K} \right)}.$$

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Problem 3 (Darboux's Proof of the Addition Theorem). Consider a simple pendulum (normalized with unit length, unit gravity, and unit mass) whose initial angle is α. The equation of motion for the angle θ of the pendulum at time u is given by

$$(\sharp) \qquad \frac{\sin \frac{\theta}{2}}{\sin \frac{\alpha}{2}} = \operatorname{sn} u,$$

where the modulus k of sn is $\sin \frac{\alpha}{2}$. Consider an identical second simple pendulum (normalized with unit length, unit gravity, and unit mass) whose initial angle is α. The equation of motion for the angle φ of the pendulum at time v is given by

$$(\flat) \qquad \frac{\sin \frac{\varphi}{2}}{\sin \frac{\alpha}{2}} = \operatorname{sn} v.$$

Let c be a constant. Assume that the two pendulums are synchronized according to the condition $u + v = c$ so that all three variables θ, φ, v are functions of u. Let $\xi = \operatorname{sn} u$ and $\eta = \operatorname{sn} v$ and consider a particle P of unit mass at the point with coordinates (ξ, η) in $\mathbb{R}^2$. Consider the following *modified angular momentum* of the particle P about the origin of $\mathbb{R}^2$

$$\frac{\eta \left(\frac{d\xi}{du} \right) - \xi \left(\frac{d\eta}{du} \right)}{1 - k^2 \xi^2 \eta^2}$$

(which differs from the usual angular momentum $\eta \left(\frac{d\xi}{du} \right) - \xi \left(\frac{d\eta}{du} \right)$ by an extra denominator). Show that

$$(\dagger) \qquad \frac{\eta \left(\frac{d\xi}{du} \right) - \xi \left(\frac{d\eta}{du} \right)}{1 - k^2 \xi^2 \eta^2} \equiv \operatorname{sn} c$$

(identically in u) by differentiating the modified angular momentum with respect to u and using the two equations of motion $(\sharp)$ and $(\flat)$ and differentiating the defining differential equation

$$\left(\frac{d}{dw} \operatorname{sn} w \right)^2 = \left(1 - \operatorname{sn}^2 w \right) \left(1 - k^2 \operatorname{sn}^2 w \right)$$

for sn. Derive from $(\dagger)$ the addition formula

$$\operatorname{sn}(u + v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$