Math 213a Homework November 5, 2004

Notations. τ is a complex number with $\operatorname{Im}\tau > 0$ and $q = e^{i\pi\tau}$.

$$\vartheta_1(w) = -i\, e^{iw+\frac{1}{4}\pi i\tau}\vartheta_4\left(w+\frac{1}{2}\pi\tau\right) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n e^{\frac{1}{4}(2n+1)^2 i\pi\tau}e^{(2n+1)iw}$$

$$= 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)w,$$

$$\vartheta_2(w) = \vartheta_1\left(w+\frac{\pi}{2}\right) = \sum_{n=-\infty}^{\infty} e^{\frac{1}{4}(2n+1)^2 i\pi\tau}e^{(2n+1)iw} = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}\cos(2n+1)w,$$

$$\vartheta_3(w) = \vartheta_4\left(w+\frac{\pi}{2}\right) = \sum_{n=-\infty}^{\infty} e^{n^2 i\pi\tau}e^{2niw} = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nw.$$

$$\vartheta_4(w) = \sum_{n=-\infty}^{\infty}(-1)^n e^{n^2 i\pi\tau}e^{2niw} = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nw.$$

Problem 1. Directly verify the following two identities.

(a)

$$\prod_{\nu=1}^{4}\vartheta_3(w_\nu) = \exp\left(\frac{1}{\pi i\tau}\sum_{\nu=1}^{4}w_\nu^2\right)\sum_{n_1,n_2,n_3,,n_4}\exp\left(\frac{1}{\pi i\tau}\sum_{\nu=1}^{4}\left(n_\nu\frac{\pi i\tau}{2}+iw_\nu\right)^2\right),$$

where each n_ν $(1\le\nu\le 4)$ runs through all *even* integers.

(b)

$$\prod_{\nu=1}^{4}\vartheta_2(w_\nu) = \exp\left(\frac{1}{\pi i\tau}\sum_{\nu=1}^{4}w_\nu^2\right)\sum_{n_1,n_2,n_3,,n_4}\exp\left(\frac{1}{\pi i\tau}\sum_{\nu=1}^{4}\left(n_\nu\frac{\pi i\tau}{2}+iw_\nu\right)^2\right),$$

where each n_ν $(1\le\nu\le 4)$ runs through all *odd* integers.

Problem 2. Let

$$A = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}$$

and

$$\begin{pmatrix} n_1' \\ n_2' \\ n_3' \\ n_4' \end{pmatrix} = A \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}.$$

(a) Verify, by using the orthogonality and symmetry of the real 4×4 matrix A, that

$$\sum_{j=1}^{4} \left(n_j'\right)^2 = \sum_{j=1}^{4} \left(n_j\right)^2$$

and

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} = A \begin{pmatrix} n_1' \\ n_2' \\ n_3' \\ n_4' \end{pmatrix}.$$

(b) Verify that when n_1, n_2, n_3, n_4 are either all even integers or all odd integers, then also n_1', n_2', n_3', n_4' are all even integers or all odd integers.

Hint: write $n_j = 4p_j + r_j$ and $n_j' = 4p_j' + r_j'$ with p_j, p_j', r_j, r' being integers and $0 \le r_j < 4$ and $0 \le r_j' < 4$.

(c) Let

$$(\dagger) \qquad \begin{pmatrix} w_1' \\ w_2' \\ w_3' \\ w_4' \end{pmatrix} = A \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}.$$

Use Part (a) and Part (b) of Problem 1 and Part (a) and Part (b) of this problem to verify that

$$\prod_{\nu=1}^{4} \vartheta_2(w_\nu) + \prod_{\nu=1}^{4} \vartheta_3(w_\nu) = \prod_{\nu=1}^{4} \vartheta_2(w_\nu') + \prod_{\nu=1}^{4} \vartheta_3(w_\nu').$$

Problem 3. Denote

$$\vartheta_\lambda(w_1)\vartheta_\mu(w_2)\vartheta_\nu(w_3)\vartheta_\rho(w_4)$$

by $(\lambda\mu\nu\rho)$ and denote

$$\vartheta_\lambda(w_1')\vartheta_\mu(w_2')\vartheta_\nu(w_3')\vartheta_\rho(w_4')$$

by $(\lambda\mu\nu\rho)'$, where w_ν and w_μ' are related by (†). Verify the following identities by translating Part (c) of Problem 2 by half-periods $\frac{\pi}{2}$ and $\frac{\pi\tau}{2}$.

$$\begin{aligned}
(3333) - (2222) &= (4444)' + (1111)' \\
(4444) + (1111) &= (3333)' - (2222)' \\
(4433) + (1122) &= (4433)' + (1122)' \\
(4433) - (1122) &= (3344)' + (2211)' \\
(3241) - (2314) &= (3241)' - (2314)'.
\end{aligned}$$

Problem 4. By using identities of the type given in Problem 3 and choosing suitable values for w_j and w_j' in terms of complex numbers x and y, verify the following addition formulae of Jacobi for theta functions.

$$\begin{aligned}
\vartheta_1(x+y)\vartheta_1(x-y)\vartheta_4(0)^2 &= \vartheta_3(x)^2\vartheta_2(y)^2-\vartheta_2(x)^2\vartheta_3(y)^2 = \vartheta_1(x)^2\vartheta_4(y)^2-\vartheta_4(x)^2\vartheta_1(y)^2, \\
\vartheta_2(x+y)\vartheta_2(x-y)\vartheta_4(0)^2 &= \vartheta_4(x)^2\vartheta_2(y)^2-\vartheta_1(x)^2\vartheta_3(y)^2 = \vartheta_2(x)^2\vartheta_4(y)^2-\vartheta_3(x)^2\vartheta_1(y)^2, \\
\vartheta_3(x+y)\vartheta_3(x-y)\vartheta_4(0)^2 &= \vartheta_4(x)^2\vartheta_3(y)^2-\vartheta_1(x)^2\vartheta_2(y)^2 = \vartheta_3(x)^2\vartheta_4(y)^2-\vartheta_2(x)^2\vartheta_1(y)^2, \\
\vartheta_4(x+y)\vartheta_4(x-y)\vartheta_4(0)^2 &= \vartheta_3(x)^2\vartheta_3(y)^2-\vartheta_2(x)^2\vartheta_2(y)^2 = \vartheta_4(x)^2\vartheta_4(y)^2-\vartheta_1(x)^2\vartheta_1(y)^2.
\end{aligned}$$