Math 213a Homework October 15, 2004

Problem 1. Verify the following infinite product expansion

$$e^{az} - e^{bz} = (a-b)\, z\, e^{\frac{1}{2}(a+b)z} \prod_{n=1}^{\infty} \left(1 + \frac{(a-b)^2 z^2}{4n^2\pi^2}\right)$$

for a, $b \in \mathbb{C}$.

Problem 2 (Stein, p.105, #13). Suppose $f(z)$ is holomorphic in a punctured disk $D_r(z_0) - \{z_0\}$ of radius r centered at z_0. Suppose also that

$$|f(z)| \leq \frac{A}{|z-a_0|^{1-\varepsilon}}$$

for some positive real numbers A and ε, and all z near z_0. Show that the singularity of f at z_0 is removable (*i.e.,* f can be extended to a holomorphic function on the entire disk $D_r(z_0)$ of radius r centered at z_0.)

Problem 3 (Stein, p.105, #14). Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.
Hint: Apply to $f\left(\frac{1}{z}\right)$ the Casorati-Weierstrass theorem [Stein, p.86, Theorem 3.3] that in any punctured disk where a holomorphic function has an essential singularity at its center (*i.e.,* the Laurent series has an infinite number of nonzero negative power terms) the image of the function must be dense in $\mathbb{C}$.

Problem 4 (Stein, p.105, #15). Use the Cauchy inequalities or the maximum modulus principle to solve the following problems.

(a) Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} |f(z)| \leq A\, R^k + B$$

for all $R > 0$, and for some integer $k \geq 0$ and some constants A, $B > 0$, then f is a polynomial of degree $\leq k$.

(b) Show that if f is holomorphic in the unit disk, is bounded, and, for some $\varphi > 0$, converges uniformly to zero in the sector $\theta < \arg z < \varphi$ as $|z| \to 1$, then $f = 0$.

(c) Let $w_1, \cdots, w_n$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that the product of the distances from z to the points w_j, $1 \le j \le n$, is at least 1. Conclude that there exists a point w on the unit circle such that the product of the distances from w to the points w_j, $1 \le j \le n$, is exactly equal to 1.

(d) Show that if the real part of an entire function f is bounded, then f is constant.

Problem 5 (Stein, p.106, #19). Prove the maximum principle for harmonic functions, that is:

(a) If u is a non-constant real-valued harmonic function in a region Ω, then u cannot attain a maximum (or a minimum) in Ω.

(b) Suppose that Ω is a region with compact closure $\bar{\Omega}$. If u is harmonic in Ω and continuous in $\bar{\Omega}$, then

$$\sup_{z\in\Omega} |u(z)| \le \sup_{z\in\bar{\Omega}-\Omega} |u(z)|.$$

Hint: To prove the first part, assume that u attains a local maximum at z_0. Let f be holomorphic near z_0 with $u = \mathrm{Re}(f)$, and show that f is open. The second part follows directly from the first.

Problem 6 (Stein, p.107, #20). This exercise shows how the mean square convergence dominates the uniform convergence of holomorphic functions on compact subsets. If U is an open subset of $\mathbb{C}$, we use the notation

$$\|f\|_{L^2(U)} = \left(\int_U |f(z)|^2 \, dxdy \right)^{\frac{1}{2}}$$

for the mean square norm, and

$$\|f\|_{L^\infty(U)} = \sup_{x\in U} |f(z)|$$

for the sup norm.

(a) If f is holomorphic in a neighborhood of the disk $D_r(z_0)$ of radius r centered at z_0, show that for any $0 < s < r$ there exists a constant $C > 0$ (which depends on s and r) such that

$$\|f\|_{L^\infty(D_s(z_0))} \le C \|f\|_{L^2(D_r(z_0))} .$$

(b) Prove that if $\{f_n\}$ is a Cauchy sequence of holomorphic functions in the mean square norm $\|\cdot\|_{L^2(U)}$, then the sequence $\{f_n\}$ converges uniformly on every compact subset of U to a holomorphic function.

Hint: Use the mean value property that the value of a holomorphic function at the center of a circle is equal to the average of its values on the circle (which is a direct consequence of the Cauchy integral formula after parametrizing the circle by the cosine and sine of the angle at the center of the circle).

Problem 7 (Another Derivation of the Poisson Integral Formula [Stein, p.109, #2]). Let u be a harmonic function in the unit disk that is continuous on its closure. Deduce Poisson's integral formula

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z_0|^2}{\left|e^{\sqrt{-1}\,\theta} - z_0\right|^2} \, u\left(e^{\sqrt{-1}\,\theta}\right) d\theta \qquad \text{for } |z_0| < 1$$

from the special case $z_0 = 0$ (the mean value theorem). Show that if $z_0 = r\, e^{\sqrt{-1}\,\varphi}$, then

$$\frac{1 - |z_0|^2}{\left|e^{\sqrt{-1}\,\theta} - z_0\right|^2} = \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2}.$$

Hint: Set $u_0(z) = u(T(z))$ where

$$T(z) = \frac{z_0 - z}{1 - \overline{z_0}\, z}.$$

Prove that u_0 is harmonic. Then apply the mean value theorem to u_0, and make a change of variables in the integral.