

## Math 213a Homework October 1, 2004

*Problem 1 (Stein, p.64, #1).* Prove that

$$\int_0^\infty \sin(x^2) dx = \int_0^\infty \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}.$$

These are Fresnel integrals. Here,  $\int_0^\infty$  is interpreted as  $\lim_{R \rightarrow \infty} \int_0^R$ .

*Hint:* Integrate the function  $e^{-z^2}$  over the path consisting of  $\{\theta = 0, 0 \leq r \leq R\}$ ,  $\{0 \leq \theta \leq \frac{\pi}{4}, r = R\}$ , and  $\{\theta = \frac{\pi}{4}, 0 \leq r \leq R\}$ , where  $z = Re^{i\theta}$ . Note that  $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$ , because its square is equal to  $\int_{(x,y) \in \mathbb{R}^2} e^{-(x^2+y^2)} dx dy$  which can be computed by changing over to polar coordinates.

*Problem 2 (Stein, p.64, #3).* Evaluate the integrals

$$\int_0^\infty e^{-ax} \cos bx dx \quad \text{and} \quad \int_0^\infty e^{-ax} \sin bx dx, \quad a > 0$$

by integrating  $e^{-\sqrt{a^2+b^2}z}$ , over an appropriate section with angle  $\cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right)$ .

*Problem 3 (Stein, p.65, #4).* Prove that for all  $\xi \in \mathbb{C}$  we have

$$e^{-\pi\xi^2} = \int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi x\xi} dx.$$

*Problem 4 (Stein, p.65, #7).* Let  $\mathbb{D}$  be  $\{z \in \mathbb{C} \mid |z| < 1\}$ . Suppose  $f : \mathbb{D} \rightarrow \mathbb{C}$  is holomorphic. Show that the diameter  $d = \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|$  of the image of  $f$  satisfies

$$2|f'(0)| \leq d.$$

Moreover, show that equality holds precisely when  $f$  is linear,  $f(z) = a_0 + a_1 z$ .

*Hint:*  $2f'(0) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2} d\zeta$  whenever  $0 < r < 1$ .

*Problem 5 (Stein, p.103, #2).* Evaluate the integral

$$\int_{-\infty}^\infty \frac{dx}{1+x^4}.$$

where the poles of  $\frac{1}{1+z^4}$ ?

*Problem 6 (Stein, p.103, #3).* Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \pi \frac{e^{-ia}}{a}, \quad \text{for } a > 0.$$

*Problem 7 (Stein, p.103, #4).* Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-ia}, \quad \text{for } a > 0.$$

*Problem 8 (Stein, p.103, #5).* Show that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{(1 + x^2)^2} dx = \frac{\pi}{2} (1 + 2\pi|\xi|) e^{-2\pi|\xi|}$$

for all  $\xi$  real.

*Problem 9 (Stein, p.104, #6).* Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

for all  $\xi$  real.

*Problem 10 (Stein, p.104, #11).* Show that if  $|a| < 1$ , then

$$\int_0^{2\pi} \log |1 - a e^{-i\theta}| d\theta = 0.$$

Then prove that the above result remains true if we assume only that  $|a| \leq 1$ .