Math 213a Homework October 22, 2004

Problem 1 (Stein, p.279, #4). Let $\omega_1, \omega_2 \in \mathbb{C}$ be $\mathbb{R}$-linearly independent. Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Lambda^* = \Lambda - \{0\}$. By rearranging the series

$$\mathfrak{P}(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right],$$

show directly, without differentiation, that $\mathfrak{P}(z+\omega) = \mathfrak{P}(z)$ whenever $\omega \in \Lambda$.

Hint: For R sufficiently large, note that

$$\mathfrak{P}(z) = \mathfrak{P}^R(z) + O\left(\frac{1}{R}\right),$$

where

$$\mathfrak{P}^R(z) := \frac{1}{z^2} + \sum_{|\omega|<R,\, \omega \in \Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right].$$

Next, observe that both $\mathfrak{P}^R(z+\omega_1) - \mathfrak{P}^R(z)$ and $\mathfrak{P}^R(z+\omega_2) - \mathfrak{P}^R(z)$ are

$$O\left(\sum_{R-c<|\omega|<R+c} \frac{1}{|\omega|^2} \right) = O\left(\frac{1}{R}\right),$$

where $c > 0$ is a constant depending on ω_1, ω_2.

Problem 2 (Stein, p.280, #7). Setting $\tau = \frac{1}{2}$ in the expression

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\tau)^2} = \frac{\pi^2}{\sin^2(\pi\tau)},$$

deduce that

$$\sum_{m \geq 1,\, m \text{ odd}} \frac{1}{m^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{m \geq 1} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

Similarly, using

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\tau)^4}$$

deduce that

$$\sum_{m\geq 1,\, m \text{ odd}} \frac{1}{m^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_{m\geq 1} \frac{1}{m^4} = \frac{\pi^4}{90}.$$

Problem 3. Let m be an integer ≥ 3 and let $a_1, \cdots, a_m$ be m distinct points in $\mathbb{C}$. Consider the compact Riemann surface M constructed so that the function $y = \left(\prod_{j=1}^m (x - a_j)\right)^{\frac{1}{2}}$ is single-valued and holomorphic. Let g be the genus of M (*i.e.*, the number of "holes" in the compact Riemann surface M).

(a) Express g in terms of m (and pay attention to the parity of m).

(b) Describe the $2g$ loops in M which form a basis of the fundamental group of M.

(c) Use the maximum principle for harmonic functions to show that there are at most g holomorphic 1-forms on M which are linearly independent over the field of complex numbers $\mathbb{C}$.

(d) Use appropriate polynomials $P(x)$ to find g holomorphic 1-forms on M of the form $P(x)\frac{dx}{y}$, with $y = \left(\prod_{j=1}^m (x - a_j)\right)^{\frac{1}{2}}$, which are linearly independent over the field of complex numbers $\mathbb{C}$.

Problem 4. Let $\mathfrak{P}(w)$ be the Weierstrass $\mathfrak{P}$ -function for the lattice $\mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ with ω_1 and ω_2 linearly independent over $\mathbb{R}$. Show that the three numbers $\mathfrak{P}(\frac{\omega_1}{2})$, $\mathfrak{P}(\frac{\omega_2}{2})$, and $\mathfrak{P}(\frac{\omega_1+\omega_2}{2})$ are distinct.

Problem 5. Let $\mathfrak{P}(w)$ be the Weierstrass $\mathfrak{P}$ -function. Let

$$\mathfrak{P}(w) = \frac{1}{w^2} + \sum_{n=2}^{\infty} c_n \frac{1}{w^{2n-2}}$$

be its Laurent series expansion about $w = 0$. Derive the following recurrent formula for the coefficients c_n of the Laurent series expansion for $n \geq 4$.

$$(n-3)(2n+1)c_n = 3 \sum_{\nu=2}^{n-2} c_\nu c_{n-\nu}.$$

Problem 6. Let $\mathfrak{P}(w)$ be the Weierstrass $\mathfrak{P}$-function for the lattice $\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ with ω_1 and ω_2 linearly independent over $\mathbb{R}$. Show that

$$\mathfrak{P}'(w)\mathfrak{P}'\left(w + \frac{\omega_1}{2}\right)\mathfrak{P}'\left(w + \frac{\omega_2}{2}\right)\mathfrak{P}'\left(w + \frac{\omega_1 + \omega_2}{2}\right) = g_2^3 - 27\, g_3^2$$

for all $w \in \mathbb{C}$, where $\mathfrak{P}'$ is the first-order derivative of $\mathfrak{P}$,

$$g_2 = 60 \sum_{\omega \in \Lambda - 0} \frac{1}{\omega^4}$$

and

$$g_3 = 140 \sum_{\omega \in \Lambda - 0} \frac{1}{\omega^6}.$$