

Math 213a Homework October 29, 2004

Notations. For a given lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_1, \omega_2 \in \mathbb{C}$ are \mathbb{R} -linearly independent, the Weierstrass P -function \mathfrak{P} for the lattice L means

$$\mathfrak{P}(w) = \frac{1}{w^2} + \sum_{\ell \in L - \{0\}} \left(\frac{1}{(w - \ell)^2} - \frac{1}{\ell^2} \right).$$

The Weierstrass sigma-function σ for the lattice L means

$$\sigma(w) = w \prod_{\ell \in L - \{0\}} \left(1 - \frac{w}{\ell} \right) e^{\frac{w}{\ell} + \frac{1}{2} \left(\frac{w}{\ell} \right)^2}.$$

The symbols $g_2 = g_2(L)$ and $g_3 = g_3(L)$ for the lattice L mean the following.

$$g_2 = g_2(L) = 60 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^4},$$
$$g_3 = g_3(L) = 140 \sum_{\ell \in L - \{0\}} \frac{1}{\ell^6}.$$

Problem 1 (Stein, p.280, #8). Let

$$E_4(\tau) = \sum_{(n,m) \neq (0,0)} \frac{1}{(n + m\tau)^4}$$

be the Eisenstein series of order 4 defined for the variable $\tau \in \mathbb{C}$ with $\text{Im } \tau > 0$.

(a) Show that $E_4(\tau) \rightarrow \frac{\pi^4}{45}$ as $\text{Im } \tau \rightarrow \infty$.

(b) More precisely,

$$\left| E_4(\tau) - \frac{\pi^4}{45} \right| \leq c e^{-2\pi t} \quad \text{if } \tau = x + \sqrt{-1}t \text{ and } t \geq 1.$$

(c) Deduce that

$$\left| E_4(\tau) - \frac{1}{\tau^4} \frac{\pi^4}{45} \right| \leq c \frac{1}{t^4} e^{-2\pi t} \quad \text{if } \tau = \sqrt{-1}t \text{ and } 0 < t \leq 1.$$

Problem 2 (Stein, p.282, #4). Suppose τ is purely imaginary, say $\tau = \sqrt{-1}t$ with $t > 0$. Consider the division of the complex plane into congruent rectangles obtained by considering the lines $x = \frac{n}{2}$, $y = \frac{tm}{2}$ as n and m range over the integers. (An example is the rectangle whose vertices are $0, \frac{1}{2}, \frac{1}{2} + \frac{\tau}{2}$, and $\frac{\tau}{2}$.)

- (a) Show that the Weierstrass P -function \mathfrak{P} for the lattice $\mathbb{Z} + \mathbb{Z}\tau$ is real-valued on all these lines, and hence on the boundaries of all these rectangles.
- (b) Prove that \mathfrak{P} maps the interior of each rectangle biholomorphically onto the upper (or lower) half-plane.

Problem 3. Let \mathfrak{P} be the Weierstrass P -function for a lattice L . Show that

$$\mathfrak{P}(z - w) - \mathfrak{P}(z + w) = \frac{\mathfrak{P}'(z)\mathfrak{P}'(w)}{[\mathfrak{P}(z) - \mathfrak{P}(w)]^2}.$$

Problem 4. Let \mathfrak{P} be the Weierstrass P -function for a lattice L . Show that

$$\mathfrak{P}(2z) = \frac{\mathfrak{P}(z)^4 + \frac{1}{2}g_2(L)\mathfrak{P}(z)^2 + 2g_3(L)\mathfrak{P}(z) + \frac{1}{16}g_2(L)^2}{4\mathfrak{P}(z)^3 - g_2(L)\mathfrak{P}(z) - g_3(L)}.$$

Problem 5. Suppose e_1, e_2, e_3 are three complex numbers. Show that a necessary and sufficient condition for the following statement to hold is that e_1, e_2, e_3 are distinct and $e_1 + e_2 + e_3 = 0$. There exist two complex numbers ω_1 and ω_2 which are \mathbf{R} -linearly independent such that the Weierstrass \mathfrak{P} function for the lattice $\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ satisfies $\mathfrak{P}(\frac{\omega_1}{2}) = e_1$, $\mathfrak{P}(\frac{\omega_2}{2}) = e_2$, and $\mathfrak{P}(\frac{\omega_1 + \omega_2}{2}) = e_3$.

Problem 6. Let τ be a complex number whose imaginary part is positive and let L be the lattice $\mathbf{Z} + \mathbf{Z}\tau$. Let

$$J(\tau) = \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2}.$$

Prove that the upper half-plane is the natural domain for the function $J(\tau)$ in the sense that it is holomorphic for τ in the upper half plane and cannot be analytically continued across any point on the real axis.

Problem 7. Let \wp be the Weierstrass P -function for a lattice L and σ be the Weierstrass σ function for L . Show that

$$\wp(z) - \wp(w) = - \frac{\sigma(z-w)\sigma(z+w)}{\sigma^2(z)\sigma^2(w)}$$

by comparing zeroes and poles of both sides.

Problem 8. Let \wp be the Weierstrass P -function for a lattice L and σ be the Weierstrass σ function for L . Show that

$$\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w) \end{vmatrix} = 2 \frac{\sigma(v-w)\sigma(w-u)\sigma(u-v)\sigma(u+v+w)}{\sigma^3(u)\sigma^3(v)\sigma^3(w)}$$

by considering the left-hand side as an elliptic function in u and looking at its poles and zeroes and using the previous exercise.

Problem 9. Let \wp be the Weierstrass P -function for a lattice L and σ be the Weierstrass σ function for L .

(a) Show that the function

$$\frac{\sigma(nz)}{\sigma(z)^{n^2}}$$

is elliptic in the sense that it is a meromorphic function on all of \mathbb{C} and L is precisely the set of all its periods.

(b) Show that

$$\frac{\sigma(2z)}{\sigma(z)^4}$$

is equal to $-\wp'(z)$.