Math 213a Homework October 8, 2004

Problem 1.

(a) (Stein, p.104, #7) Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{\frac{3}{2}}}, \quad \text{whenever } a > 1.$$

(b) (Stein, p.104, #9) Show that

$$\int_0^1 \log(\sin \pi x)\, dx = -\log 2$$

by integration along the boundary of

$$\{0 < x < 1, 0 < y < \infty\}.$$

(c) (Stein, p.104, #10) Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2+a^2}\, dx = \frac{\pi}{2a} \log a$$

by integrating along the boundary of

$$\{y > 0,\ \varepsilon < |z| < R\}.$$

(d) Show that, if $-1 < a < 1$ and $0 < \beta < \pi$,

$$\int_0^\infty \frac{x^a\, dx}{1+2x\cos\beta+x^2} = \frac{\pi \sin a\beta}{\sin a\pi \sin\beta}.$$

Problem 2. Verify the following infinite partial fraction expansions of the given meromorphic functions on $\mathbb{C}$ (some of which are written with pairs of terms for indices and their negatives combined).

(a)

$$\sec z = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n \left(n+\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)^2 \pi^2 - z^2}.$$

(b)

$$\tan z = 2z \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2 \pi^2 - z^2}.$$

(c)

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{z^2 + 4n^2\pi^2}.$$

(d)

$$\operatorname{cosec}^2 z = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n\pi)^2}.$$

Problem 3. Let a be any complex number with $n^4 + a^4 \neq 0$ for any positive integer n. Sum the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + a^4}$$

and

$$\sum_{n=1}^{\infty} \frac{n^2}{n^4 + a^4}.$$

Problem 4. By applying the Argument Principle to the domain which is the intersection of a quadrant and the disk centered at the origin of radius R with $R \to \infty$, show that two of the roots of the following equation lie in the second quadrant and the other two in the the third quadrant.

$$z^4 + z^3 + 4z^2 + 2z + 3 = 0.$$

Problem 5. Use the Argument Principle to prove the following. Let C be a simple closed piecewise smooth curve in $\mathbb{C}$ and $f(z)$ be a holomorphic function defined on C and on the domain enclosed by C. Suppose $f(z)$ has no zero on C. Let k be a positive integer. Suppose the number of points of C where the real part of $f(z)$ vanishes is $2k$ (without counting multiplicities). Then $f(z)$ has at most k zeroes inside C.

Problem 6. Use Rouché's theorem to prove the following theorem of Hurwitz. Suppose D is a domain (*i.e.*, a connected open set) in $\mathbb{C}$. Let $f_n(z)$ be a sequence of holomorphic function on D approaching a *nonconstant* holomorphic function $f(z)$ on D *uniformly* as $n \to \infty$. If $f_n : D \to \mathbb{C}$ is injective for each n, show that $f : D \to \mathbb{C}$ is injective.

Problem 7 (Stein, p.108, #1). Consider a nonconstant holomorphic map on the unit disk $f : \mathbb{D} \to \mathbb{C}$ which satisfies $f(0) = 0$. By the open mapping theorem, the image $f(\mathbb{D})$ contains a small disk centered at the origin. We then ask: does there exist $r > 0$ such that for *all* $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$, we have $D_r(0) \subset f(\mathbb{D})$? Here $D_r(0)$ means the open disk in $\mathbb{C}$ centered at the origin with radius r.

(a) Show that with no further restrictions on f, no such r exists. It suffices to find a sequence of nonconstant functions $\{f_n\}$ holomorphic in $\mathbb{D}$ such that $\frac{1}{n} \notin f(\mathbb{D})$. Compute $f_n'(0)$ and discuss.

(b) Assume in addition that f also satisfies $f'(0) = 1$. Show that despite this new assumption, there exists no $r > 0$ satisfying the desired condition.

Hint: Try $f_\varepsilon(z) = \varepsilon\left(e^{\frac{z}{\varepsilon}} - 1\right)$.

The Koebe-Bieberbach theorem states that if in addition to $f(0) = 0$ and $f'(0) = 1$ we also assume that f is injective, then such an r exists and the best possible value is $r = \frac{1}{4}$.

(c) As a first step, show that if

$$h(z) = \frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$

is holomorphic and injective on $0 < |z| < 1$, then $\sum_{n=1}^{\infty} n\,|c_n|^2 < 1$.

Hint: Calculate the area of the complement of $h\left(D_\rho(0) - \{0\}\right)$ where $0 < \rho < 1$, and let $\rho \to 1$.

(d) If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ satisfies the hypotheses of the theorem, show that there exists another function g satisfying the hypotheses of the theorem such that $(g(z))^2 = f\left(z^2\right)$.

Hint: $\frac{f(z)}{z}$ is nowhere vanishing so there exists ψ such that $(\psi(z))^2 = \frac{f(z)}{z}$ and $\psi(0) = 1$. Check that $g(z) = z\psi(z^2)$ is injective.

(e) With the notation of the previous part, show that $|a_2| \leq 2$, and that equality holds if and only if

$$f(z) = \frac{z}{\left(a - e^{\sqrt{-1}\,\theta} z\right)^2} \quad \text{for some } \theta \in \mathbb{R}.$$

Hint: What is the power series expansion of $\frac{1}{g(z)}$? Use part (c).

(f) If $h(z) = \frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$ is injective on $\mathbb{D}$ and avoids the values z_1 and z_2, show that $|z_1 - z_2| \leq 4$.

Hint: Look at the second coefficient in the power series expansion of $\frac{1}{h(z) - z_j}$.

(g) Complete the proof of the theorem.

Hint: If f avoids w, then $\frac{1}{f}$ avoids 0 and $\frac{1}{w}$.