PROOF OF THE BIEBERBACK CONJECTURE

(3.1) *Statement of the Bieberbach Conjecture.* Let S be the class of all univalent holomorphic functions $f(z)$ on the open unit 1-disk $\mathbb{D}$ normalized with $f(0) = 0$ and $f'(0) = 1$. The power series expansion of $f(z)$ centered at $z = 0$ is of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

The Bieberback conjecture states that $|a_n| \leq n$ for all $n \geq 2$.

Before the proof let us say something about its rough idea first. The main idea is more or less like the continuity method using an evolution equation, which in our case is the Löwner differential equation. We approximate the given function by a special class of function, namely those which maps the open unit disk biholomorphically to $\mathbb{C}$ minus a curved ray (which starts at some point of $\mathbb{C}$ and goes to ∞ along a simple smooth curve.

For any member of the special class of functions we automatically get a 1-parameter family of them by using the Riemann mapping theorem to map the open unit disk to $\mathbb{C}$ minus a shortened ray which starts not from the beginning but from some point in the ray. This family satisfies a differential equation derived from the Poisson kernel (or more precisely the Schwarz kernel which includes the value of the imaginary part of the holomorphic function at the origin), called the Löwner differential equation. The moduli for members of the special class of functions are given by certain unit-circle-valued functions of a real variable. We want to use the continuity method and the integration of the differential equation to show that certain inequality called the Lebedev-Milin inequaltiy, which implies the Bieberbach conjecture, persists in the family when the family approaches our original given function.

The way to get this is to compare the situation with the standard family which is the deformation of the Koebe extremal function

$$z \mapsto \frac{z}{(1-z)^2}$$

(which maps the open unit disk to $\mathbb{C} - \left[\frac{1}{4}, \infty\right)$ and is the composite of a Möbius transformation and the map

$$z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right)$$

used in defining the cosine function from the rotated exponential function.) A member of the standard family is the Riemann mapping which maps the open unit disk biholomorphically onto $\mathbb{C} - [s, \infty)$ for some $\frac{1}{4} \leq s < \infty$. The standard family after the parameter is appropriately renormalized is given by $z \mapsto w(z,t)$ satisfying the equation

$$\frac{z}{(1-z)^2} = \frac{e^t w}{(1-w)^2}.$$

The rôle played by this standard family is more or less like that of the barrier function or what is used in a comparison theorem.

The idea is to push as far as possible the technique of roots of the function to make the image of the new function avoid certain subsets of $\mathbb{C}$. It means using the logarithm of the function. The Legendre functions and their addition theorems turn out to provide crucial inequalities of numbers from generating functions needed for the comparison.

We now start with the proof the Bieberbach conjecture. First of all we observe that without loss of generality we can assume that $f(z)$ is defined in some open neighborhood of the topological closure $\overline{\mathbb{D}}$ of $\mathbb{D}$. To see this, we need only replace $f(z)$ by $\frac{1}{r} f(rz)$ for $r < 1$ and then let $r \to 1^-$. Let Ω be the image of $\mathbb{D}$ under f so that f can be extended continuously to a map from $\overline{\mathbb{D}}$ to $\overline{\Omega}$.

(3.2) *Reduction to Biholomorphisms Between the Unit Disk and the Complement of a Simple Curve Segment.* Join a point P_0 on the boundary of Ω to infinity by a smooth simple curve Γ in $\mathbb{C} - \Omega$. We now start from P_0 and parametrize the boundary $\partial\Omega$ of Ω by $t \mapsto \psi(t)$ for $0 \leq t \leq \ell$ in the counterclockwise sense so that $\psi(\ell) = \psi(0) = P_0$. We now parametrize the curve Γ from the point P_0 to infinity by $t \mapsto \psi(t)$ for $\ell \leq t < \infty$. For $0 \leq s < \infty$, let C_s be the curve given by $t \mapsto \psi(t)$ for $s \leq t < \infty$.

Note that for $t > 0$ the domain $\mathbb{C} - C_t$ is simply connected, but the domain $\mathbb{C} - C_0 = \Omega \cup \left(\mathbb{C} - \overline{\Omega}\right)$ consists of the two disjoint components Ω and $\mathbb{C} - \overline{\Omega}$.

For $0 < t < \infty$, let $g(z,t)$ be the univalent holomorphic map from $\mathbb{D}$ onto $\mathbb{C} - C_t$ so that $g(0,t) = 0$ and

$$\beta(t) := \frac{\partial g}{\partial z}(0,t) > 0.$$

For fixed $0 < t \leq t_0$ the image of $g(z,t)$ as a function of z is contained in $\mathbb{C} - C_{t_0}$. Since $\mathbb{C} - C_{t_0}$, is simply connected and is a proper domain in $\mathbb{C}$, it follows from Riemann mapping theorem that $\mathbb{C} - C_{t_0}$ is biholomorphic to the open unit 1-disk $\mathbb{D}$ under some $\Phi : \mathbb{C} - C_{t_0} \to \mathbb{D}$ mapping the origin to the origin. The composite $\Phi\left(g(z,t)\right)$ of $g(z,t)$ with Φ as a function of z is a normal family with the parameter $0 < t \leq t_0$ for the family. For every sequence t_ν in $(0, t_0)$ there exists a subsequence which either tends uniformly to a constant function with value at the boundary of $\mathbb{D}$ or to a holomorphic function from $\mathbb{D}$ to $\mathbb{D}$. (This is because of the maximum modulus principle applied to $|z| \circ (\Phi(g(z,t)))$.) Since the family maps the origin to the origin, the former case cannot occur. Then $g(z,t)$ converges uniformaly on compact subsets of $\mathbb{D}$ to some univalent holomorphic function $\tilde{f}(z)$ from $\mathbb{D}$ to Ω as $t \to 0$. We still have to show that the image of $\tilde{f}(z)$ covers all of Ω. We will use the $\frac{1}{4}$ theorem of Koebe and the univalent Kobayashi metric d_t for $\mathbb{C} - C_t$ defined by univalent maps to make sure that the following holds (see the appendix on the $\frac{1}{4}$ theorem of Koebe and the univalent Kobayashi metric). When we join the origin to any prescribed point Q in Ω by a path γ, the path γ has univalent Kobayashi metric $d_t(\gamma)$ uniformly bounded in t, because the path γ stays a fixed Euclidean distance away from the boundary of $\mathbb{C} - C_t$ for all $t \geq 0$. We can now take the inverse $\tilde{\gamma}_t$ of γ with respect to $g(z,t)$ and it is contained in a fixed compact subset of $\mathbb{D}$ independent of $t \geq 0$. This implies that Q is in the image of $\tilde{f}(z)$. Because of normalization that

$$\left.\frac{\partial g(z,t)}{\partial z}\right|_{z=0} > 0,$$

it follows that both the derivatives of $f'(z)$ and $f(z)$ with respect to z is positive at $z = 0$ and, having the same image, both functions $\tilde{f}(z)$ and $f(z)$ are the same. So $g(z,t)$ approaches $f(z)$ as $t \to 0+$. Thus, it suffices to prove the Bieberback conjecture for $g(z,t)$ for $t > 0$.

(3.3) *Reparametrization of Simple Curve Segment.* By Schwarz's lemma, we know that $\beta(t)$ is a strictly increasing function of t. ($\beta(t)$ can be interpreted as the Kobayashi metric defined by univalent maps and monotonicity for domains holds for it.) We are going to reparametrize the curve C_0 (*i.e.,* by composing the function $t \mapsto \psi(t)$ with a diffeomorphism $\tau \mapsto t(\tau)$ of the interval $[0, T)$ to $[0, \infty)$) so that $\beta(t(\tau)) = e^\tau$ and, after replacing $g(z,t)$ by $g(z, t(\tau))$ and t by τ, ther derivative $\beta(t) = \frac{\partial g}{\partial z}(0,t)$ at the origin is equal to

e^t. In other words, the curve C is parametrized by

$$\log \frac{\partial g}{\partial z}(0,t)$$

when $z \mapsto g(z,t)$ is the Riemann mapping from $\mathbb{D}$ to $\mathbb{C}$ minus that part of C from the point labelled by t.

After this reparametrization the range of t is $[0,T]$ and it may happen that $T < \infty$. If indeed $T < \infty$, then for any fixed positive number M, there exists $\epsilon_M > 0$ such that C_t is outside $\{|w| \leq M\}$ for $T - \epsilon_M < t < T$. This means that the image of $g(z,t)$ contains $\{|w| \leq M\}$ for $T - \epsilon_M < t < T$. The inverse of $g(z,t)$ maps $\{|w| \leq M\}$ to $\mathbb{D}$. This means that for $r > 0$ sufficiently close to 1 the image of $\{|z| = r\}$ under $g(z,t)$ must be outside $\{|w| \leq M\}$. We now apply the maximum modulus principle to the holomorphic function $\frac{z}{g(z,t)}$ on $\{|z| = r\}$ and let $r \to 1-$ to conclude that

$$\left|\frac{z}{g(z,t)}\right| \leq \frac{1}{M}$$

for $z \in \mathbb{D}$ and $T - \epsilon_M < t < T$. In particular, at $z = 0$ we get

$$M \leq \left|\frac{\partial g}{\partial z}(0,t)\right| = e^t$$

for $T - \epsilon_M < t < T$. This means that $e^T \geq M$ for any fixed positive number M, which is a contradiction. After this reparametrization we have

$$g\left(z,t\right) = e^t \left(z + \sum_{n=2}^{\infty} a_n(t) z^n\right). \tag{3.3.1}$$

(3.4) *Löwner's Differential Equation.* We are going to derive the Löwner differential equation which is a first-order differential equation involving the partial derivatives of $g(z,t)$ with respect to z and t and some unspecified function $\kappa(t)$ of absolute value identically 1. Before we start the derivation of the Löwner differential equation, we introduce a change of notations to have a new setting for it.

Now that we have defined $g(z,t)$ whose image is $\mathbb{C} - C_t$ and know the normailization of its power series (3.3.1), we forget our original function $f(z)$ on $\mathbb{D}$ and Ω, after the arbitrary choice of some $t_0 > 0$ (which eventually will be allowed to approach zero), use the notation $f(z)$ for $e^{-t_0} g(z,t_0)$ and change $g(z,t)$ to $e^{-t_0} g\left(z, t + t_0\right)$.

Our setup now is as follows. We have a simple smooth curve C in $\mathbb{C} - 0$ going from a point P_0 to ∞ parametrized by $\psi : [0, \infty) \to \mathbb{C}$ and for every $\leq t < \infty$ there is a function

$$g(z, t) = e^t \left(z + \sum_{n=2}^{\infty} a_n(t) z^n \right)$$

which maps $\mathbb{D}$ biholomorphically onto $\mathbb{C} - C_t$. Let $\lambda(t)$ be the boundary point of $\mathbb{D}$ which is mapped by (the extension of) $g(z, t)$ to the point $\psi(t) \in C_t \subset C$. Let $f(z) = g(z, 0)$.

We now start the derivation of the Löwner differential equation. Let $G(z, t)$ be the inverse function of $g(z, t)$. Let

$$f(z, t) = G(f(z), t) = e^{-t} \left(z + \sum_{n=2}^{\infty} b_n(t) z^n \right).$$

Then $f(z, t)$ maps $\mathbb{D}$ to $\mathbb{D}$ minus a slit from a boundary point to its interior. For $0 < s < t$, let $h(z, s, t) = G(g(z, s), t)$ which maps $\mathbb{D}$ to $D - J_{s,t}$ for some curve $J_{s,t}$ which starts at the point $\lambda(t)$ in the boundary of $\mathbb{D}$ and goes into $\mathbb{D}$.

Note that $\lambda(t) = G(\psi(t), t)$ and the curve $J_{s,t}$ is equal to the (closure of the) image of $C_s - C_t$ under $G(z, s)$, but this piece of information is not important to us at this point. We denote by $B_{s,t}$ the part of the boundary of $\mathbb{D}$ which corresponds to $J_{s,t}$ under $h(z, s, t)$. Then $B_{s,t}$ is equal to the (closure of the) image of $C_s - C_t$ under (the extension of) $G(z, s)$.

Let $e^{i\alpha}$ and $e^{i\beta}$ be the end-points of $B_{s,t}$ so that $e^{i\alpha} = G(\psi(s), s) = \lambda(s)$ and $e^{i\beta} = G(\psi(t), s)$.

When we consider $g(z, t)$ as a function which labels the point $g(z, t)$ in $\mathbb{C} - C_t$ by the label $z \in \mathbb{D}$, the function $h(z, s, t)$ tells us the label at later time t of the point which is labelled as z at the earlier time s. One very important point (which is the key point in the derivation of the Löwner differential equation is that $h(z, s, t)$ maps $\partial\mathbb{D} - B_{s,t}$ to $\partial\mathbb{D} - \{\lambda(t)\}$ and maps the interior of $B_{s,t}$ to the curve $J_{s,t} - \{\lambda(t)\}$ inside $\mathbb{D}$.

We now let

$$\Phi(z) = \Phi(z, s, t) = \log \left(\frac{h(z, s, t)}{z} \right).$$

(This step, as well as the later step of differentiation, imitates the process of taking logarithmic derivative of

$$\frac{z}{(1-z)^2} = \frac{e^t w}{(1-w)^2}$$

to derive its Löwner equation

$$\frac{\partial w}{\partial t} = -w\,\frac{1-w}{1+w}.$$

Another reason for this step is that the vanishing of $\operatorname{Re}\Phi$ on $\partial\mathbb{D} - B_{s,t}$ from $h(z,s,t)$ mapping $\partial\mathbb{D} - B_{s,t}$ to $\partial\mathbb{D} - \{\lambda(t)\}$.) We have

$$\operatorname{Re}\Phi = \log\left|\frac{h(z,s,t)}{z}\right|$$

and $\Phi(0,s,t) = s - t$, because

$$g(z,s) = e^s\left(z + \sum_{n=2}^{\infty} b_n(t) z^n\right)$$

and

$$G(z,t) = e^{-t}\left(z + \sum_{n=2}^{\infty} B_n(t) z^n\right)$$

so that

$$h(z,s,t) = e^{s-t}\left(z + \sum_{n=2}^{\infty} c_n(t) z^n\right).$$

Thus

$$\operatorname{Re}\Phi(z) = 0 \text{ on } \partial\mathbb{D} - B_{s,t},$$
$$\operatorname{Re}\Phi(z) < 0 \text{ on } B_{s,t} - \partial B_{s,t}.$$

Since the imaginary part of Φ vanishes at 0, the Schwarz integral formula (which is the Poisson formula including the value of the imaginary part of the holomorphic function at the origin) gives

$$\Phi(z) = \frac{1}{2\pi}\int_{\theta=\alpha}^{\beta} \operatorname{Re}\Phi\left(e^{i\theta}\right)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta. \tag{3.4.1}$$

At $z = 0$ the above formula yields

$$s - t = \Phi(0) = \frac{1}{2\pi}\int_{\theta=\alpha}^{\beta} \operatorname{Re}\Phi\left(e^{i\theta}\right)\, d\theta. \tag{3.4.2}$$

From the Mean Value Theorem of Calculus applied separately to the real part and imaginary part we obtain $\alpha \le \sigma \le \beta$ and $\alpha \le \tau \le \beta$ such that

(3.4.3)

$$\begin{aligned} &\log\frac{h(z,s,t)}{z} \\ &= \frac{1}{2\pi}\left(\operatorname{Re}\left(\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\right) + \sqrt{-1}\operatorname{Im}\left(\frac{e^{i\tau}+z}{e^{i\tau}-z}\right)\right)\int_{\theta=\alpha}^{\beta}\operatorname{Re}\Phi\left(e^{i\theta}\right)\, d\theta \\ &= \frac{1}{2\pi}\left(\operatorname{Re}\left(\frac{e^{i\sigma}+z}{e^{i\sigma}-z}\right) + \sqrt{-1}\operatorname{Im}\left(\frac{e^{i\tau}+z}{e^{i\tau}-z}\right)\right)(s-t), \end{aligned}$$

where for the last identity (3.4.2) is used.

Now divide (3.4.3) by $s - t$ and pass to limit as s approaches t. (This step is just using the limit of the difference quotient to differentiate.) Then the arc B_{st} approaches the point $\lambda(t)$, forcing both $e^{i\sigma}$ and $e^{i\tau}$ to approach $\lambda(t)$, and we get

$$\left.\frac{\partial}{\partial s}\log\frac{h(z,s,t)}{z}\right|_{s=t} = \frac{\lambda(s)+z}{\lambda(s)-z}. \tag{3.4.4}$$

Now we use the chain rule to compute

$$\begin{aligned} \left.\frac{\partial}{\partial s}\log\frac{h(z,s,t)}{z}\right|_{s=t} &= \left.\frac{1}{h(z,s,t)}\frac{\partial}{\partial s}h(z,s,t)\right|_{s=t} \\ = \left.\frac{1}{h(z,s,t)}\frac{\partial}{\partial s}G\left(g(z,s),t\right)\right|_{s=t} &= \left.\frac{1}{h(z,s,t)}\frac{\partial G}{\partial z}(g(z,s),t)\frac{\partial g}{\partial s}(z,s)\right|_{s=t} \\ = \left.\frac{1}{h(z,s,t)}\frac{1}{\frac{\partial g}{\partial z}h(z,s,t)}\frac{\partial g}{\partial s}(z,s)\right|_{s=t} &= \frac{\frac{\partial g}{\partial t}(z,t)}{z\frac{\partial g}{\partial z}(z,t)}, \end{aligned}$$

because $h(z,t,t) = z$. Let $\kappa(t) = \frac{1}{\lambda(t)}$. Now (3.4.4) reads

$$\frac{\partial g}{\partial t}(z,t) = z\,\frac{\partial g}{\partial z}(z,t)\,\frac{1+\kappa(t)z}{1-\kappa(t)z} \tag{3.4.5}$$

with $|\kappa(t)| \equiv 1$, which is the Löwner differential equation.

We could interpret the Löwner differential equation as the equation of motion of a system which is biholomorphically parametrized by $\mathbb{D}$ with normalization at the $0 \in \mathbb{D}$ so that at time t the system precisely fills out completely $\mathbb{C} - C_t$. The function $g(z,t)$ gives the position of the point $z \in \mathbb{D}$ at time t.

(3.5) *Lebedev-Milin Inequality.* Let

$$\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n z^n.$$

We are going to show by an inequality of Lebedev-Milin that the Bieberbach conjecture is a consequence of the inequality

$$\sum_{k=1}^{n} \left(\frac{4}{k} - k \left| c_k \right|^2 \right) (n-k+1) \geq 0$$

for every $n \geq 1$.

The Lebedev-Milin inequality simply tells us how the Bieberbach conjecture $|a_n| \leq n$ is translated into in terms of c_n through the correspondence

$$\log \frac{z + \sum_{n=1}^{\infty} a_n z^n}{z} = \sum_{n=1}^{\infty} c_n z^n.$$

Note that, when $f(z)$ is the Koebe extremal function

$$\frac{z}{(1-z)^2},$$

we have

$$\log \frac{f(z)}{z} = -2 \log(1-z) = \sum_{n=1}^{\infty} \frac{2}{n} z^n$$

and $c_n = \frac{2}{k}$ and the first factor

$$\frac{4}{k} - k \left| c_k \right|^2$$

in the summand of the Lebedev-Milin inequality simply vanishes. So the Koebe extremal function gives also the extremal case in the Lebedev-Milin inequality.

Heuristically, this Lebedev-Milin inequality is related to the transformation which sends the unit disk to the right half-plane. This idea of replacing absolute-value bounds by nonnegativity permeates the entire proof of the Bieberbach conjecture, as we will see later from the use of the Poisson formula, the Löwner differential equations, and the deformation of the special Koebe extremal function. The transformation

$$w \mapsto z = \frac{w-1}{w+1}$$

sends the right half-plane to the unit disk. Its inverse is

$$w = \frac{1+z}{1-z}$$

which sends the unit disk to the right half-plane, as we can also see from

$$w = \frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1+z-\bar{z}-|z|^2}{|1-z|^2}$$

and

$$\operatorname{Re} w = \frac{1-|z|^2}{|1-z|^2}.$$

Note that the factor on the right-hand side

$$\frac{1+\kappa(t)z}{1-\kappa(t)z}$$

of Löwner's differential equation has real part nonnegative for $|z| < 1$, because $|\kappa(t)| = 1$ and the transformation

$$w = \frac{1+z}{1-z}$$

sends the open unit disk to the open right half-plane.

To prove this Lebedev-Milin inequality, we introduce

$$h(z) = \sqrt{f\left(z^2\right)} = \sum_{n=1}^{\infty} b_{2n-1} z^{2n-1}. \tag{3.5.1}$$

(Note that this is essentially the same transformation which gives the Bieberbach conjecture $|a_2| \leq 2$ for the coefficient of the square term, as given in the appendix below on the $\frac{1}{4}$-theorem of the Koebe. Here the relations between the coefficients of the original function and those of the new functions are given completely instead of the small part enough to give $|a_2| \leq 2$. Also, instead of getting an inequality by using the square root $\sqrt{f(z)}$ to make the image miss a fixed open subset of $\mathbb{C}$, one could make a better inequality by using a higher-order root $\sqrt[n]{f(z)}$ for a large positive integer n to miss a bigger fixed open subset of $\mathbb{C}$. The ultimate use of this trick is to consider $\log f(z)$. The use of $\sqrt{f(z^2)}$ instead of $\sqrt{f(z)}$ is a technical step to make the new function well-defined. It is the same technical reason to use $\log \frac{f(z)}{z}$ instead of $\log f(z)$.) The set of coefficients $\{b_n\}$ serves as a bridge between $\{a_n\}$ and $\{c_n\}$.

We express the coefficients of the new function h in terms of the coefficients c_n and get

$$\frac{h\left(\sqrt{z}\right)}{\sqrt{z}} = \sum_{n=0}^{\infty} b_{2n+1} z^n = \exp\left(\sum_{n=1}^{\infty} \frac{c_n}{2} z^n\right). \tag{3.5.2}$$

We use the renormalization $\beta_n = b_{2n+1}$ and $\alpha_n = \frac{c_n}{2}$ to rewrite the last part of (3.5.2) as

$$\sum_{n=0}^{\infty} \beta_n z^n = \exp\left(\sum_{n=1}^{\infty} \alpha_n z^n\right). \tag{3.5.3}$$

(Our goal is to inequalities for the coefficients a_n from inequalities for the coefficients c_n and for this goal we go through the intermediate step of using inequalities for the coefficients b_n.) Differentiating (3.5.3) to get

$$\sum_{n=1}^{\infty} n\beta_n z^{n-1} = \left(\sum_{\ell=1}^{\infty} \ell\alpha_\ell z^{\ell-1}\right) \exp\left(\sum_{n=1}^{\infty} \alpha_n z^n\right) = \left(\sum_{\ell=1}^{\infty} \ell\alpha_\ell z^{\ell-1}\right)\left(\sum_{n=0}^{\infty} \beta_n z^n\right)$$

and equating the coefficients of like powers of z yield

$$\beta_n = \frac{1}{n} \sum_{k=0}^{n-1} (n-k)\alpha_{n-k}\, \beta_k.$$

By Cauchy's inequality

$$\beta_n^2 \leq \frac{1}{n^2}\left(\sum_{k=1}^{n} k^2 \left|\alpha_k\right|^2\right)\left(\sum_{k=0}^{n-1} \left|\beta_k\right|^2\right).$$

Let

$$A_n = \sum_{k=1}^{n} k^2 \left|\alpha_k\right|^2,$$

$$B_n = \sum_{k=0}^{n} \left|\beta_k\right|^2.$$

Then

$$\begin{aligned} B_n = B_{n-1} + \left|\beta_n\right|^2 \leq \left(1 + \frac{1}{n^2} A_n\right) B_{n-1} \\ = \frac{n+1}{n}\left(1 + \frac{A_n - n}{n(n+1)}\right) B_{n-1} \leq \frac{n+1}{n} B_{n-1} \exp\left(\frac{A_n - n}{n(n+1)}\right). \end{aligned}$$

(This step simply uses the above Cauchy's inequality to dominate B_n by expressions in A_n. We are led to taking the exponential of the expression in A_n, because when we inductively apply the inequality to B_{n-1} and other B_ν with $\nu = n-2, n-3, \cdots, 1$ we get a product formula in A_n. The technical change of n^2 to $n(n-1)$ is to facilitate later summation in n using the telescopic process.) Inductively, we obtain

$$B_n \leq (n+1) \exp\left(\sum_{k=1}^{n} \frac{A_k - k}{k(k+1)}\right)$$

whose right-hand side can be rewritten as

$$(n+1) \exp\left(\sum_{k=1}^{n} \frac{A_k}{k(k+1)} + 1 - \sum_{k=1}^{n+1} \frac{1}{k}\right).$$

Let

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

and we use summation by parts to rewrite

$$\begin{aligned} \sum_{k=1}^{n} \frac{A_k}{k(k+1)} &= \sum_{k=1}^{n} A_k \left(s_k - s_{k-1}\right) \\ &= A_n s_n - \sum_{k=1}^{n} \left(A_k - A_{k-1}\right) s_{k-1} \end{aligned}$$

$$= \left(\sum_{k=1}^{n} k^2 \left|\alpha_k\right|^2\right) s_n - \sum_{k=1}^{n} k^2 \left|\alpha_k\right|^2 s_{k-1}$$
$$= \sum_{k=1}^{n} k^2 \left|\alpha_k\right|^2 \left(\frac{n}{n+1} - \frac{k-1}{k}\right)$$
$$= \sum_{k=1}^{n} \frac{k(n+1-k)}{n+1} \left|\alpha_k\right|^2.$$

Thus we have

$$B_n \leq (n+1) \exp\left(\frac{1}{n+1} \sum_{k=1}^{n} (n+1-k) \left(k \left|\alpha_k\right|^2 - \frac{1}{k}\right)\right).$$

This means that
(3.5.4)
$$\sum_{k=0}^{n} \left|b_{2k+1}\right|^2 \leq (n+1) \exp\left(\frac{1}{4(n+1)} \sum_{k=1}^{n} (n+1-k) \left(k \left|c_k\right|^2 - \frac{4}{k}\right)\right).$$

By squaring both sides of (3.5.1) and equating the coefficients of like powers of z, we obtain

$$a_n = b_1 b_{2n-1} + b_3 b_{2n-3} + \cdots + b_{2n-1} b_1$$

and by Cauchy's inequality

(3.5.5)
$$\left|a_n\right| \leq \left(\left|b_1\right|^2 + \left|b_3\right|^2 + \cdots + \left|b_{2n-1}\right|^2\right)^{\frac{1}{2}} \left(\left|b_{2n-1}\right|^2 + \left|b_{2n-3}\right|^2 + \cdots + \left|b_1\right|^2\right)^{\frac{1}{2}}$$
$$= \left|b_1\right|^2 + \left|b_3\right|^2 + \cdots + \left|b_{2n-1}\right|^2.$$

If one has the inequality

$$\sum_{k=1}^{n} \left(\frac{4}{k} - k \left|c_k\right|^2\right) (n-k+1) \geq 0,$$

then (3.5.4) would imply that

$$\sum_{k=0}^{n} \left|b_{2k+1}\right|^2 \leq n+1$$

and (3.5.5) would imply that

$$|a_n| \leq n.$$

This finishes the verification of the Lebedev-Milin inequality.

(3.6) *Reduction to Inequalities Involving Legendre Functions by Using the Koebe Extremal Function and Its Deformation, the Fundamental Theorem of Calculus, and the Löwner Differential Equation.* The extremal Köebe function is the function

$$\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z_n.$$

(This special exremal Köebe function maps the open unit disk to $\mathbb{C}$ minus the slit $[\frac{1}{4}, \infty)$, as one can see by using polar coordinates to see that $z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps the open unit disk to $\mathbb{C}$ minus the slit $[-1, 1]$ and then applying a Möbius transformation.)

We introduce the deformation $w(z) = w_t(z)$ for $0 \leq t < \infty$ which maps $\mathbb{D}$ to $\mathbb{D}$ minus a curve segment from the boundary of $\mathbb{D}$ to its interior. The initial part of the power series expansion of function $w_t(z)$ is $e^{-t}z$ and the defining equation is

$$\frac{z}{(1-z)^2} = \frac{e^t w}{(1-w)^2}. \tag{3.6.1}$$

The extremal family $w(z, t) = w_t(z)$ satisfies the differential equation

$$\frac{\partial w}{\partial t} = -w \frac{1-w}{1+w},$$

as one can easily check as follows. (This equation is the same as the Löwner differential equation in another form with $\kappa \equiv -1$, as given in an appendix below, in which the Möbius transformation on the right-hand side in the variable z is replaced by one in the function w and the replacement absorbs the partial derivative of the function w with respect to the variable z.) Take logarithm of both sides of (3.6.1) to get

$$\log \frac{z}{(1-z)^2} = t + \log w - 2\log(1-w)$$

and differentiate with respect to t to get

$$0 = 1 + \left(\frac{1}{w} - 2\frac{1}{w-1}\right)\frac{\partial w}{\partial t}$$

which gives

$$\frac{\partial w}{\partial t} = \frac{-1}{\frac{1}{w} - 2\frac{1}{w-1}} = \frac{-w(w-1)}{w-1-2w} = -w\,\frac{1-w}{1+w}.$$

Write

$$\log\left(\frac{g(z,t)}{e^t z}\right) = \sum_{k=1}^{\infty} c_k(t) z^k \tag{3.6.2}$$

so that the Bieberbach conjecture follows from

$$\sum_{k=1}^{n}\left(\frac{4}{k} - k\left|c_k(0)\right|^2\right)(n-k+1) \geq 0$$

for every $n \geq 1$. We now put this sequence of inequalities into the coefficients of a generating function and consider

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\frac{4}{k} - k\left|c_k(0)\right|^2\right)(n-k+1)\right) z^{n+1}$$

and proceed to prove that all the coefficients in this power series are nonnegative. We are going to reduce this power series to some expression involving the Legendre functions. The reduction process uses

(i) the switching of the order of summation,

(ii) using the Koebe extremal function,

(iii) using the fundamental theorem of calculus,

(iv) Löwner's differential equation for the deformation $w(z,t)$ of the Koebe extremal function,

(v) the formula for the coefficients of the Fourier series to express $c_k'(t)$ as an integral,

(vi) Cauchy's theorem for holomorphic functions, and

(vii) the Löwner differential equation for $g(z,t)$.

The technique of using an integral expression to give an estimate is analogous to the use of the Cauchy integral formula for derivatives to estimate the coefficients in a power series expansion of a holomorphic function. The Cauchy integral formula is an application of the Stokes's theorem for dimension two, which is a generalization of the fundamental theorem of calculus. Here we will use both the Cauchy integral formula for derivatives and the fundamental theorem of calculus to get our integral expression. Our integral expression enables us to use the Löwner differential equation and the comparison to the deformation of the Koebe extremal function.

To carry out this reduction, we first express $c_k'(t)$ as an integral by the Cauchy integral formula for derivatives. Differentiating (3.6.2) with respect to t gives

$$\frac{g_t}{g} - 1 = \sum_{k=1}^{\infty} c_k'(t) z^k$$

and

$$c_k'(t) = \lim_{r \to 1^-} \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \frac{\partial_t g(z_1, t)}{g(z_1, t)} \, \bar{z}_1^k d\theta \tag{3.6.3}$$

where $z_1 = re^{i\theta}$. Differentiating (3.6.2) with respect to z gives

$$\frac{g_z}{g} - \frac{1}{z} = \sum_{k=1}^{\infty} k \, c_k(t) z^{k-1}$$

which can rewritten as

$$\frac{g}{z g_z} \left(1 + \sum_{\ell=1}^{\infty} \ell \, c_\ell(t) z^\ell \right) = 1. \tag{3.6.4}$$

We now start the reduction process which will involve all the steps listed above.

$$\begin{aligned} &\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \left(\frac{4}{k} - k \left| c_k(0) \right|^2 \right) (n-k+1) \right) z^{n+1} \\ &= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \left(\frac{4}{k} - k \left| c_k(0) \right|^2 (n-k+1) \right) z^{n+1} \\ &\qquad \text{(change order of summation)} \end{aligned}$$

$$= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{4}{k} - k\left|c_k(0)\right|^2\right) m z^{m+k}$$

(relabel index)

$$= \frac{z}{(1-z)^2}\sum_{k=1}^{\infty}\left(\frac{4}{k} - k\left|c_k(0)\right|^2\right) z^k$$

(sum to Koebe function)

$$= \int_{t=0}^{\infty} -\frac{z}{(1-z)^2}\frac{d}{dt}\sum_{k=1}^{\infty}\left(\frac{4}{k} - k\left|c_k(t)\right|^2\right) w^k$$

(fundamental theorem of calculus,
using $w(z,0)=z$ and $w(z,t)\to 0$ as $t\to\infty$)

$$= \int_{t=0}^{\infty}\frac{e^t w}{1-w^2}\frac{1+w}{1-w}\left(\sum_{k=1}^{\infty} k\left(c_k(t)\bar{c}_k(t)\right)' w^k + \sum_{k=1}^{\infty}\left(4 - k^2\left|c_k(t)\right|^2\right) w^k\frac{1-w}{1+w}\right) dt$$

$$\left(\text{use the Löwner equation } \frac{\partial w}{\partial t} = -w\frac{1-w}{1+w} \text{ and definition of } w\right)$$

$$= \int_{t=0}^{\infty}\frac{e^t w}{1-w^2}\left[\frac{1+w}{1-w}\left(1+\sum_{k=1}^{\infty}\left(\lim_{r\to 1^-}\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\frac{\partial_t g(z_1,t)}{g(z_1,t)} k\bar{c}_k(t)\bar{z}_1^k d\theta\right) w^k\right)\right.$$

$$+\frac{1+w}{1-w}\left(1+\sum_{k=1}^{\infty}\left(\lim_{r\to 1^-}\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\frac{\partial_t \bar{g}(z_1,t)}{\bar{g}(z_1,t)} k c_k(t) z_1^k d\theta\right) w^k\right)$$

$$\left. -2\left(\frac{1+w}{1-w}\right) + \frac{4w}{1-w} - \sum_{k=1}^{\infty} k^2\left|c_k(t)\right|^2 w^k\right] dt$$

$$\left(\text{use (3.6.3), cancel the two terms 1 by } -2\left(\frac{1+w}{1-w}\right) \text{ and use } \frac{1}{1-w} = \sum_{k=1}^{\infty} w^k\right)$$

$$= \int_{t=0}^{\infty}\frac{e^t w}{1-w^2}\Bigg[$$

$$1+\sum_{k=1}^{\infty}\left(\lim_{r\to 1^-}\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\frac{\partial_t g(z_1,t)}{g(z_1,t)}\left(2\left(1+\bar{c}_1(t)\bar{z}_1+\cdots+k\bar{c}_k(t)\bar{z}_1^k\right) - k\bar{c}_k(t)\bar{z}_1^k\right) d\theta\right) w^k$$

$$+1+\sum_{k=1}^{\infty}\left(\lim_{r\to 1^-}\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\frac{\partial_t \bar{g}(z_1,t)}{\bar{g}(z_1,t)}\left(2\left(1+c_1(t)z_1+\cdots+kc_k(t)z_1^k\right) - kc_k(t)z_1^k\right) d\theta\right) w^k$$

$$-2-\sum_{k=1}^{\infty}k^2\left|c_k(t)\right|^2 w^k\Bigg]\,dt$$

$$\left(\text{where } \frac{1+w}{1-w}\left(1+\sum_{k=1}^{\infty}\gamma_k w^k\right)=\left(-1+2\sum_{\ell=0}^{\infty}w^\ell\right)\left(1+\sum_{k=1}^{\infty}\gamma_k w^k\right)\right.$$

$$=1+\sum_{k=1}^{\infty}\left(-\gamma_k+2\left(1+\sum_{\ell=1}^{k}\gamma_\ell\right)\right)w^k \text{ is used}$$

$$\left.\text{and } -2\frac{1+w}{1-w}+4\frac{2}{1-w}=-2 \text{ is also used}\right)$$

$$=\int_{t=0}^{\infty}\frac{e^t w}{1-w^2}\Bigg[$$

$$1+\sum_{k=1}^{\infty}\left(\lim_{r\to 1^-}\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\frac{\partial_t g(z_1,t)}{g(z_1,t)}\frac{g(z_1,t)}{z_1\partial_z g(z_1,t)}\left(1+\sum_{\ell=1}^{\infty}\ell c_\ell(t)z_1^\ell\right)\cdot\right.$$

$$\left.\cdot\left(2\left(1+\bar c_1(t)z_1+\cdots+k\bar c_k(t)\bar z_1^k\right)-k\bar c_k(t)\bar z_1^k\right)d\theta\right)w^k$$

$$+1+\sum_{k=1}^{\infty}\left(\lim_{r\to 1^-}\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\frac{\partial_t \bar g(z_1,t)}{\bar g(z_1,t)}\frac{\bar g(z_1,t)}{\bar z_1\overline{\partial_z g(z_1,t)}}\left(1+\sum_{\ell=1}^{\infty}\ell \bar c_\ell(t)\bar z_1^\ell\right)\cdot\right.$$

$$\left.\cdot\left(2\left(1+c_1(t)z_1+\cdots+kc_k(t)z_1^k\right)-kc_k(t)z_1^k\right)d\theta\right)w^k$$

$$-2-\sum_{k=1}^{\infty}k^2\left|c_k(t)\right|^2 w^k\Bigg]\,dt \quad (\text{use}(3.6.4))$$

$$=\int_{t=0}^{\infty}\frac{e^t w}{1-w^2}\left(\sum_{k=1}^{\infty}\lim_{r\to 1^-}\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\operatorname{Re}\left(\frac{1+\kappa(t)z_1}{1-\kappa(t)z_1}\right)\cdot\right.$$

$$\left.\cdot\left|2\left(1+c_1(t)z_1+\cdots+kc_k(t)z_1^k\right)-kc_k(t)z_1^k\right|^2 d\theta\right)w^k dt$$

$$\left(\text{use } \int_{\theta=0}^{2\pi}e^{ik\theta}d\theta=0 \text{ for } k\neq 0 \text{ and } \frac{\partial_t g(z_1,t)}{g(z_1,t)}\frac{g(z_1,t)}{z_1\partial_z g(z_1,t)} \text{ being a power}\right.$$

$$\left.\text{series in } z_1 \text{ and then use Löwner's equation } \frac{\partial_t g}{z\partial_z g}=\frac{1+\kappa(t)z}{1-\kappa(t)z}\right)$$

$$=\int_{t=0}^{\infty}\frac{e^t w}{1-w^2}\left(\sum_{k=1}^{\infty}A_k(t)w^k\right)dt,$$

where

$$A_k(t)=\lim_{r\to 1^-}\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\mathrm{Re}\left(\frac{1+\kappa(t)z_1}{1-\kappa(t)z_1}\right)\cdot$$
$$\cdot\left|2\left(1+c_1(t)z_1+\cdots+kc_k(t)z_1^k\right)-kc_k(t)z_1^k\right|^2 d\theta$$

which is nonnegative, because

$$\mathrm{Re}\left(\frac{1+\kappa(t)z_1}{1-\kappa(t)z_1}\right)\geq 0.$$

For the step where $\int_{\theta=0}^{2\pi}e^{ik\theta}d\theta=0$ for $k\neq 0$ and Löwner's differential equation are used, the following identity (which is simply a completion of absolute-value square) is also used.

$$\left(1+\sum_{\ell=1}^{k}\ell c_\ell(t)z_1^\ell\right)\left(2\Big(1+\bar c_1(t)z_1+\cdots+k\bar c_k(t)\bar z_1^k\Big)-k\bar c_k(t)\bar z_1^k\right)$$
$$=\frac{1}{2}\left|2\Big(1+c_1(t)z_1+\cdots+kc_k(t)z_1^k\Big)-kc_k(t)z_1^k\right|^2$$
$$+kc_k(t)z_1^k\Big(1+\bar c_1(t)z_1+\cdots+(k-1)\bar c_{k-1}(t)\bar z_1^{k-1}\Big)+\frac{1}{2}k^2\left|c_k(t)\right|^2 r^{2k}.$$

In particular, what is used in that step is

$$\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\frac{\partial_t g(z_1,t)}{g(z_1,t)}\frac{g(z_1,t)}{z_1\partial_z g(z_1,t)}d\theta=1,$$
$$\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\frac{\partial_t g(z_1,t)}{g(z_1,t)}\frac{g(z_1,t)}{z_1\partial_z g(z_1,t)}z_1^k\bar z_1^\ell d\theta=0\quad\text{if } k>\ell.$$

(Note that the contribution of the last term $\frac{1}{2}k^2\left|c_k(t)\right|^2 r^{2k}$ of the above identity precisely cancels out the contribution of the last term $-\sum_{k=1}^{\infty}k^2\left|c_k(t)\right|^2 w^k$ in the expression in the immediately preceding step.)

To summarize what we have obtained by applying the fundamental theorem of calculus to the deformation of normalized univalent maps, Löwner's differential equation, and the comparison with the deformation of Koebe's extremal function, we restate the result.

Final Form of Transformed Inequality.

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\frac{4}{k}-k\left|c_k(0)\right|^2\right)(n-k+1)\right)z^{n+1}$$
$$=\int_{t=0}^{\infty}\frac{e^t w}{1-w^2}\left(\sum_{k=1}^{\infty}A_k(t)w^k\right)dt,$$

where

$$A_k(t)=\lim_{r\to 1^-}\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\mathrm{Re}\left(\frac{1+\kappa(t)z_1}{1-\kappa(t)z_1}\right)\cdot$$
$$\cdot\left|2\left(1+c_1(t)z_1+\cdots+kc_k(t)z_1^k\right)-kc_k(t)z_1^k\right|^2 d\theta$$

and $z_1=r\,e^{i\theta}$.

Final Step of Proof of Bieberbach's Conjecture. To finish the proof of the Bieberbach conjecture, it suffices to show that all the coefficients $\Lambda_k^n(t)$ of

$$\frac{e^t w^{k+1}}{1-w^2}=\sum_{n=0}^{\infty}\Lambda_k^n(t)z^{n+1}$$

are nonnegative for all $k\geq 1$. This step does not involve the arbitrary normalized univalent function on $\mathbb{D}$ given at the very beginning. It involves only some special functions.

The nonnegativity of $\Lambda_k^n(t)$ will be verified by using Legendre functions. We first roughly explain why Legendre functions play a role in this step and then give the details later. With the identity

$$\frac{1-w^2}{1-2w\cos\theta+w^2}=1+2\sum_{n=1}^{\infty}w^n\cos n\theta$$

from the Poisson kernel (to be verified below) and with x set to be $1-e^{-t}+e^{-t}\cos\theta$, the expression

$$\frac{z}{1-2xz+z^2}$$

is transformed to

$$\frac{e^t w}{1-w^2}\left(1+2\sum_{k=1}^{\infty}w^k\cos k\theta\right)$$

and the Legendre functions $P_n(x)$ are generated by

$$\frac{1}{\sqrt{1-2xz+z^2}}=\sum_{n=0}^{\infty}P_n(x)z^n.$$

The problem is reduced to the question whether all the coefficients of

$$\frac{z}{1-2xz+z^2}$$

as a series in z^n and $\cos k\theta$ are nonnegative. Since the quotient used for the generation of the Legendre functions is

$$\frac{1}{\sqrt{1-2xz+z^2}}$$

and we are interested in its square, an addition formula for the Legendre functions will be used.

(3.7) *Use of Legendre Functions and Their Addition Theorem.* Before we look at the use of the Legendre functions, we first give an identity from the Poisson kernel. For $z=re^{i\theta}$ we have

$$\begin{aligned}1+2\sum_{n=1}^{\infty}r^n\cos n\theta &= 1+\sum_{n=1}^{\infty}r^ne^{in\theta}+\sum_{n=1}^{\infty}r^ne^{-in\theta}=1+2\,\mathrm{Re}\sum_{n=1}^{\infty}z^n\\ &=2\,\mathrm{Re}\,\frac{z}{1-z}+1=2\,\mathrm{Re}\,\left(\frac{z}{1-z}-\frac{1}{2}\right)=\mathrm{Re}\frac{1+z}{1-z}=\mathrm{Re}\frac{(1+z)(1-\bar z)}{|1-z|^2}\\ &=\mathrm{Re}\frac{(1+z-\bar z-|z|^2)}{|1-z|^2}=\frac{1-|z|^2}{|1-z|^2}=\frac{1-r^2}{1-2r\cos\theta+r^2}.\end{aligned}$$

The identity we need is

$$\frac{1-w^2}{1-2w\cos\theta+w^2}=1+2\sum_{n=1}^{\infty}w^n\cos n\theta$$

when in the preceding identity of real-analytic functions the real variable r is replaced by the complex variable w. The Legendre functions $P_n(x)$ can be defined by the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n.$$

They can also be defined by Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n.$$

The Legendre functions are special cases of a more general class of functions called the *associated Legendre functions* defined by

$$P_n^m(x) = \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

They arise in the method of separation of variables in the solution of the Laplace equation in $\mathbf{R}^3$ in spherical coordinates in the factor for the latitude variable. What we need is the following addition formula for the Legendre functions.

$$P_n\left(xx' + \sqrt{1-x^2}\sqrt{1-x'^2}\cos\theta\right)$$
$$= P_n(x) P_n(x') + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(x) P_n^m(x') \cos m\theta.$$

We apply the addition formula for the Legendre functions to the special case

$$x = x' = \sqrt{1 - e^{-t}}$$

so that

$$\sqrt{1-x^2} = \sqrt{1-x'^2} = e^{\frac{-t}{2}}$$

and

$$xx' + \sqrt{1-x^2}\sqrt{1-x'^2}\cos\theta = 1 - e^{-t} + e^t \cos\theta.$$

The addition formula for this special case reads

$$P_n\left(1 - e^t + e^t\cos\theta\right)$$
$$= \left|P_n\left(\sqrt{1-e^{-t}}\right)\right|^2 + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \left|P_n^m\left(\sqrt{1-e^{-t}}\right)\right|^2 \cos m\theta.$$

This means that all the coefficients $A_{m,n}$ of $P_n\left(1 - e^t + e^t\cos\theta\right)$ in its cosine series

$$P_n\left(1 - e^t + e^t\cos\theta\right) = \sum_{m=0}^{n} A_{m,n} \cos m\theta$$

are nonnegative.

We now link the nonnegativity of $\Lambda_k^n(t)$ to the nonnegativity of $A_{m,n}$. From the definition

$$\frac{z}{(1-z)^2}=\frac{e^t w}{(1-w)^2}$$

we have

$$\frac{(1-z)^2}{z}=\frac{(1-w)^2}{e^t w}$$

or

$$\frac{1}{z}-2+z=e^{-t}\left(\frac{1}{w}+w-2\right).$$

Set $x=1-e^{-t}+e^{-t}\cos\theta$. Then $\cos\theta=e^t x+1-e^t$ and

$$\frac{z}{1-2xz+z^2}=\frac{1}{z+\frac{1}{z}-2x}=\frac{1}{2+e^{-t}\left(\frac{1}{w}+w-2\right)-2x}$$

$$=\frac{w}{2w+e^{-t}\left(1+w^2-2w\right)-2xw}=\frac{w\,e^t}{2e^t w+1+w^2-2w-2xe^t w}$$

$$=\frac{w\,e^t}{1+w^2-2\left(e^t x+1-e^t\right)w}=\frac{e^t w}{1-2w\cos\theta+w^2}$$

$$=\frac{e^t w}{1-w^2}\,\frac{1-w^2}{1+w^2-2w\cos\theta}=\frac{e^t w}{1-w^2}\left(1+2\sum_{k=1}^{\infty}w^k\cos k\theta\right)$$

$$=\sum_{k=0}^{\infty}\Lambda_0^n(t)z^{n+1}+2\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\Lambda_k^n(t)z^{n+1}\cos k\theta.$$

Now we express

$$\frac{z}{1-2xz+z^2}$$

in terms of the Legendre functions via their definition from the generating function.

$$\frac{z}{1-2xz+z^2}=z\left(\sum_{n=0}^{\infty}P_n(x)z^n\right)^2$$

$$=z\left(\sum_{n=0}^{\infty}P_n(1-e^{-t}+e^{-t}\cos\theta)z^n\right)^2$$

$$= z \left(\sum_{n=0}^{\infty} z^n \sum_{m=0}^{n} A_{m,n} \cos m\theta \right)^2$$

$$= z \sum_{n,q=0}^{\infty} z^{n+q} \sum_{m=0}^{n} \sum_{p=0}^{q} A_{p,q} A_{m,n} \cos m\theta \cos p\theta$$

$$\frac{z}{2} \sum_{n,q=0}^{\infty} z^{n+q} \sum_{m=0}^{n} \sum_{p=0}^{q} A_{p,q} A_{m,n} \left(\cos(m-p)\theta + \cos(m+p)\theta \right).$$

Thus

$$\frac{z}{1 - 2xz + z^2} = \sum_{n=0}^{\infty} z^{n+1} \sum_{k=0}^{n} B_{k,n} \cos k\theta$$

with $B_{k,n} \geq 0$. Finally from

$$\Lambda_0^n = B_{0,n},$$
$$\Lambda_k^n = \frac{1}{2} B_{k,n} \quad (k \geq 1),$$

it follows that $\Lambda_k^n \geq 0$ for $0 \leq k \leq n$ and $n \geq 0$, which finishes the proof of the Bieberback conjecture.

Appendix on Alternative Form of Löwner's Equation

The Löwner differential equation given above is for the family

$$g(z,t)=e^{t}\left(z+\sum_{n=2}^{\infty}a_n(t)z^n\right)$$

which univalently maps the unit 1-disk $\mathbb{D}$ onto $\mathbb{C}$ minus a Jordan curve-segment C_t from some finite point to infinity. That is the Löwner differential equation which we need for the proof of the Bieberbach conjecture. There is another form of the Löwner differential equation for the family

$$f(z,t)=e^{-t}\left(z+\sum_{n=2}^{\infty}b_n(t)z^n\right)$$

which univalently maps $\mathbb{D}$ onto $\mathbb{D}$ minus a slit from a boundary point to its interior. The Löwner differntial equation for the family $f(z,t)$ is not needed for the proof of the Bieberbach conjecture. However, it provides the context for the differential equation

$$\frac{\partial w}{\partial t}=-w\frac{1-w}{1+w}$$

with $\kappa\equiv -1$ of the deformation $w(z,t)=w_t(z)$ of the Koebe extremal function. The Löwner differential equation for $f(z,t)$ is

$$\frac{\partial}{\partial t}f(z,t)=-f(z,t)\,\frac{1+\kappa(t)f(z,t)}{1-\kappa(t)f(z,t)}$$

with $|\kappa(t)|\equiv 1$. Its derivation is completely analogous to that for $g(z,t)$. The difference being one substitution of the variable z by $f(z,t)$.

From the definition of $h(z,s,t)=G\left(g(z,s),t\right)$ it follows that

$$\begin{aligned}h\left(f(z,s),s,t\right)&=G\left(g\left(f(z,s),s\right),t\right)\\&=G\left(g\left(G(f(z),s),s\right),t\right)\\&=G\left(f(z),t\right)=f(z,t).\end{aligned}$$

In (3.4.1) we replace z by $f(z,s)$ and get from (3.4.3)

$$\log\frac{f(z,t)}{f(z,s)}=\frac{1}{2\pi}\int_{\theta=\alpha}^{\beta}\operatorname{Re}\Phi\left(e^{i\theta}\right)\,\frac{e^{i\theta}+f(z,s)}{e^{i\theta}-f(z,s)}\,d\theta.$$

From the Mean Value Theorem of Calculus applied separately to the real part and imaginary part we obtain $\alpha \leq \sigma \leq \beta$ and $\alpha \leq \tau \leq \beta$ such that

$$(\dagger) \qquad \log \frac{f(z,t)}{f(z,s)}$$

$$= \frac{1}{2\pi}\left(\operatorname{Re}\left(\frac{e^{i\sigma}+f(z,s)}{e^{i\sigma}-f(z,s)}\right)+\sqrt{-1}\operatorname{Im}\left(\frac{e^{i\tau}+f(z,s)}{e^{i\tau}-f(z,s)}\right)\right)\int_{\theta=\alpha}^{\beta}\operatorname{Re}\Phi\left(e^{i\theta}\right)\,d\theta$$

$$= \frac{1}{2\pi}\left(\operatorname{Re}\left(\frac{e^{i\sigma}+f(z,s)}{e^{i\sigma}-f(z,s)}\right)+\sqrt{-1}\operatorname{Im}\left(\frac{e^{i\tau}+f(z,s)}{e^{i\tau}-f(z,s)}\right)\right)(s-t),$$

where for the last identity (3.4.2) is used. Now divide (†) and pass to limit as t approaches s. Then the arc B_{st} approaches the point $\lambda(s)$ and we get

$$\frac{\partial}{\partial s}\log f(z,s) = -\frac{\lambda(s)+f(z,s)}{\lambda(s)-f(z,s)}.$$

Let $\kappa(t) = \frac{1}{\lambda(t)}$. We have the Löwner differential equation

$$\frac{\partial}{\partial t}f(z,t) = -f(z,t)\,\frac{1+\kappa(t)f(z,t)}{1-\kappa(t)f(z,t)},$$

where $|\kappa(t)| \equiv 1$.

If we continue with the interpretation of the Löwner differential equation as an equation of motion, this alternative form of the Löwner differential equation describes the label $f(z,t)$ of the particle at time t which occupies the same position as the particle labelled by z at time 0.

The deformation $w(z,t)$ of the Koebe extremal function defined by

$$\frac{z}{(1-z)^2} = \frac{e^t w}{(1-w)^2}$$

is a special case of $f(z,t)$ and the function $\kappa(t)$ for this special case is the function which is identically -1. The alternative form of the Löwner differential equation provides the context to put the differential equation

$$\frac{\partial w}{\partial t} = -w\frac{1-w}{1+w}.$$

Appendix on Koebe's Distortion Theorem, 1/4-Theorem of Koebe, and Univalent Kobayashi Metric

Koebe's Distortion Theorem. Suppose f is a univalent holomorphic function defined on the open unit disk $\Delta = \left\{z \in \mathbf{C} \,\middle|\, |z| < 1\right\}$ which is normalized with $f(0) = 0$ and $f'(0) = 1$. Then for $|z| < 1$ one has the following inequalities

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}$$
$$\frac{|z|}{(1 + |z|)^2} \leq |f(z)| \leq \frac{|z|}{(1 - |z|)^2}.$$

This distortion theorem tells us that for a univalent function normalized at the center of the open unit disk, as one goes to the boundary from the center, the change or the distortion of the value of the function and its derivative can be estimated. The idea of the proof is the following. To get an estimate of $f(z)$ and $f'(z)$, it suffices to get a good estimate of $f''(z)$ and then integrate from the origin. When the image of $f(z)$ misses an open subset, we can get some estimate of $f(z)$. The larger the open set is, the better the estimate. One way to make the image of the function $f(z)$ miss an open subset is to take the square root $\sqrt{f(z)}$ of $f(z)$. When we take a branch the image of the other branch will be missed when f is univalent. So the technique of taking the square root makes the function miss an open set as large as the image itself. At the origin we have trouble getting a branch of $\sqrt{f(z)}$. To avoid this difficulty we consider instead a branch of $\sqrt{f(z^2)}$. We want to apply the surface area theorem. So we have to use a univalent holomorphic function which maps a neighborhood of infinity to a neighborhood of infinity. Let

$$f(z) = z + a_2 z^2 + \cdots$$

be the power series expansion. We consider the function

$$\frac{1}{\sqrt{f(\frac{1}{z^2})}} = z - \frac{a_2}{2}\frac{1}{z} + \cdots.$$

By the surface area theorem (that the inequality

$$\sum_{\nu=1}^{\infty} \nu |c_\nu|^2 \leq 1$$

holds for any map of the form

$$z \mapsto z + \sum_{\nu=0}^{\infty} \frac{c_\nu}{z^\nu}$$

which maps $\{|z| > 1\}$ biholomorphically onto an open neighborhood of the infinity point), we conclude that $|a_2| \leq 2$. This is the first inequality of the conjecture of Bieberbach. It means that $|f''(0)| \leq 4$ for any univalent holomorphic function $f(z)$ on the open unit disk which is normalized at the origin. To get an estimate of $|f''(z)|$ for $z \neq 0$, we need only to use a Möbius transformation. The Möbius transformation mapping the origin to z_0 is

$$z \longrightarrow \frac{z + z_0}{1 + z\overline{z_0}}.$$

We have to normalize the function after the Möbius transformation. So we consider the funciton

$$\frac{f\left(\frac{z+z_0}{1+z\overline{z_0}}\right) - f(z_0)}{(1 - |z_0|^2) f'(z_0)}$$

whose second derivative at the origin is

$$\frac{f''(z_0)}{f'(z_0)}(1 - |z_0|^2) - 2\overline{z_0}.$$

So we have

$$\left| \frac{f''(z_0)}{f'(z_0)}(1 - |z_0|^2) - 2\overline{z_0} \right| \leq 4.$$

We want to convert this inequality to an inequality involving real quantities. To make $2\overline{z_0}$ real, we need only multiply it by z_0. So we have

$$\left| z_0 \frac{f''(z_0)}{f'(z_0)}(1 - r^2) - 2\, r^2 \right| \leq 4\, r,$$

where $r = |z_0|$. Divide both sides by $1 - r^2$ to get rid of the factor $1 - r^2$ in

$$z_0 \frac{f''(z_0)}{f'(z_0)}(1 - r^2)$$

to facilitate integration with respect to z_0, we get

$$\left| z_0 \frac{f''(z_0)}{f'(z_0)} - \frac{2\, r^2}{1 - r^2} \right| \leq \frac{4\, r}{1 - r^2}.$$

We have no way of knowing when

$$z_0 \frac{f''(z_0)}{f'(z_0)}$$

is real. So we use its real part and get

$$\frac{4\, r}{1-r^2} - \frac{2\, r^2}{1-r^2} \leq \operatorname{Re}\left(z_0 \frac{f''(z_0)}{f'(z_0)}\right) \leq \frac{4\, r}{1-r^2} + \frac{2\, r^2}{1-r^2}.$$

We observe that

$$z \frac{f''(z_0)}{f'(z_0)} = z \frac{d}{dz} \log f'(z) = \frac{d \log f'(z)}{d \log z}.$$

When we differentiate a holomorphic function with respect to a holomorphic variable, the result is the same as differentiating the function with respect to the real part of the variable. Now the real part of the variable $\log z$ is $\log |z|$. Hence

$$z_0 \frac{f''(z_0)}{f'(z_0)}$$

is the same as

$$\frac{\partial \log \ f'(z_0)}{\partial \log r}$$

which is equal to

$$\frac{1}{r} \frac{\partial \log f'(z_0)}{\partial r}.$$

Its real part is

$$\frac{1}{r} \frac{\partial \log |f'(z_0)|}{\partial r}.$$

So we get

$$-2\, \frac{2-r}{1-r^2} \leq \frac{\partial \log |f'(z_0)|}{\partial\, r} \leq 2 \frac{2+r}{1-r^2}.$$

Integrating from the origin we get

$$\frac{1-r}{(1+r)^3} \leq |f'(z_0)| \leq \frac{1+r}{(1-r)^3}$$

and

$$\frac{r}{(1+r)^2} \leq |f(z_0)| \leq \frac{r}{(1-r)^2}.$$

As a consequence of the distortion theorem we conclude that the set of all univalent holomorphic functions on a Riemann surface M which are normalized at some point P_0 of M up to order one is a normal family. Here normalized means that for some local coordinate system z centered at P_0 the value of the function at P_0 is zero and its derivative with respect to z is 1.

Corollary ($\frac{1}{4}$-Theorem of Koebe). Let $f(z)$ be a univalent holomorphic function on the unit 1-disk $\mathbb{D}$, normalized by $f(0) = 0$ and $f'(0) = 1$. Then the image $f(\mathbb{D})$ of f contains the open disk of radius $\frac{1}{4}$ centered at the origin. The number $\frac{1}{4}$ is sharp, because the example $f = \frac{z}{(1+e^{i\alpha}z)^2}$ maps $\mathbb{D}$ biholomorphically onto $\mathbb{C}$ minus $\left\{z = re^{-i\alpha} \,|\, \frac{1}{4} \leq r < \infty\right\}$.

Remark. If one does not need the sharp bound $\frac{1}{4}$, one can more easily get a lower bound $\rho_0 > 0$ with a simple normal family argument as follows. First the set $\mathcal{F}$ of all univalent holomorophic function $f(z)$ on $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$ form a normal family. The reason is as follows. Let $a = a_f$ be the point with the smallest absolute value in the complement of the image of f. By Schwarz lemma applied to the inverse of f on $|z| < a$, we conclude from $f(0) = 0$ and $f'(0) = 1$ that $|a| \leq 1$. Let $g = g_f = \frac{f}{a}$ so that the point with the smallest absolute value in the complement of the image of g is always 1. Let $h = h_f$ be the branch of $\sqrt{g(z) - 1}$ so that its value at 0 is i. Then its image omits the domain Ω which is the image of $\mathbb{D}$ under the branch $z \mapsto \sqrt{z-1}$ with value $-i$ at 0. Thus the set $\{h_f \,|\, f \in \mathcal{F}\}$ whose elements are holomorphic functions from $\mathbb{D}$ to $\mathbb{C} - \Omega$ is a normal family. It follows from $a_f \leq 1$ and $f = a_f \left(h_f^2 + 1\right)$ that $\mathcal{F}$ is a normal family. To prove the existence of $\rho_0 > 0$ so that the image of f contains $|z| < \rho_0$ for $f \in \mathcal{F}$, we assume the contrary that $a_{f_\nu} \to 0$ as $\nu \to \infty$ for some sequence $f_\nu \in \mathcal{F}$. Since $\{h_f \,|\, f \in \mathcal{F}\}$ is normal, by replacing $\{\nu\}$ by a subsequence we can assume without loss of generality that h_{f_ν} converges to a univalent holomorphic function on $\mathbb{D}$ uniformly on compacdt subsets of $\mathbb{D}$ (the case of convergence to a constant function being ruled out by $h_{f_\nu}(0) = 0$). From $f_\nu = a_{f_\nu} \left(h_{f_\nu}^2 + 1\right)$ and $a_{f_\nu} \to 0$ as $\nu \to \infty$ we conclude that $f_\nu \to 0$ uniformly on compact subsets of $\mathbb{D}$, which contradicts the normalization $f_\nu'(0) = 1$.

Kobayashi Metric Defined by Univalent Functions. Let Ω be an open subset of $\mathbb{C}$. For $P \in \Omega$ define the *Univalent Kobayashi* metric $|\cdot|_\Omega$ of the tangent space at P as follows. For a tangent vector ξ of Ω at P, define $|\xi|$ be the smallest $\frac{1}{a}$ such that there exists a holomorphic univalent map f from $\{\, z \in \mathbb{C} \,|\, |z| < a \,\}$

to Ω so that $f(0) = P$ and df maps $\frac{\partial}{\partial z}$ at the origin of $\mathbb{C}$ to ξ. This "univalent Kobayashi metric" is clearly invariant under biholomorphic maps. For a point P of Ω let $d_\Omega(P)$ be the Euclidean distance to the boundary of Ω. By the $\frac{1}{4}$-theorem of Koebe,

$$\frac{1}{4\, d_\Omega(P)} \leq \left|\left(\frac{\partial}{\partial z}\right)_P\right|_\Omega \leq \frac{1}{d_\Omega(P)}.$$

After we integrate the *infinitesimal* univalent Kobayashi metric with respect to a path, we have the the following two conclusions.

(a) Ω is complete with respect to its univalent Kobayashi metric, because of

$$\left|\left(\frac{\partial}{\partial z}\right)_P\right|_\Omega \geq \frac{1}{4\, d_\Omega(P)}$$

and

$$\int_{t=1}^{\infty} \frac{dt}{t} = \infty.$$

(b) Let P_1 and P_2 be any two points of Ω which can be joined by a path C of Euclidean length $\ell_C > 0$ so that the Euclidean distance from any point of C to the boundary of Ω is at least $\delta_C > 0$. Then the univalent Kobayashi distance between P_1 and P_2 with respect to Ω is no more than $\frac{\ell_C}{\delta_C}$, because

$$\left|\left(\frac{\partial}{\partial z}\right)_P\right|_\Omega \leq \frac{1}{d_\Omega(P)}.$$