Theorem of Borel-Caratheodory

Statement. Let $f(z)$ be a holomorphic function on $|z| \leq R$, and let $M(r) = \sup_{|z|=r} |f(z)|$ and $A(r) = \sup_{|z|=r} \operatorname{Re} f(z)$. Then for $0 < r < R$,

$$M(r) \leq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|.$$

Proof. Assume first that $f(0) = 0$ and f is nonconstant. Then $A(R) > A(0) = 0$. Let

$$\phi(z) = \frac{f(z)}{2A(R) - f(z)}.$$

Then $\phi(z)$ is holomorphic on $|z| \leq R$, because the real part of the denominator does not vanish on $|z| \leq R$. We know $\phi(0) = 0$. Let $f = u + iv$. Then

$$|\phi(z)|^2 = \frac{u^2 + v^2}{(2A(R) - u)^2 + v^2} \leq 1,$$

because

$$-2A(R) + u \leq u \leq 2A(R) - u.$$

By Schwarz's lemma,

$$|\phi(z)| \leq \frac{r}{R}.$$

Hence

$$|f(z)| = \left| \frac{2A(R)\phi(z)}{1 + \phi(z)} \right| \leq \frac{2A(R)r}{R-r},$$

which finishes the proof under the additional assumption of $f(0)$ and $f(z)$ not identically constant.

Now we consider the case where $f(0) \neq 0$. We apply the preceding result to $f(z) - f(0)$ instead of $f(z)$ and get

$$|f(z) - f(0)| \leq \frac{2r}{R-r} \sup_{|z|=R} \operatorname{Re}(f(z) - f(0)) \leq \frac{2r}{R-r} (A(R) + |f(0)|)$$

and

$$|f(z)| \leq \frac{2r}{R-r} (A(R) + |f(0)|) + |f(0)| = \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|.$$

Q.E.D.

Remark. We can also get a similar result by using the Poisson integral formula (or more precisely the Schwarz integral formula).

Landau's Lemma. Suppose $f(s)$ is a holomorphic function on $|s-s_0|\leq r$ and $M>1$ such that

$$\left|\frac{f(s)}{f(s_0)}\right|<e^M.$$

Let Z be the set of all zeroes of $f(s)$ on $|s-s_0|\leq\frac{r}{2}$ with multiplicities counted. Then

$$\left|\frac{f'(s)}{f(s)}-\sum_{\rho\in Z}\frac{1}{s-\rho}\right|<\frac{48M}{r}$$

on $|s-s_0|\leq\frac{r}{4}$.

Proof. Let

$$g(s)=\frac{f(s)}{\prod_{\rho\in Z}(s-\rho)}.$$

Then $g(s)$ is holomorphic on $|s-s_0|\leq r$ and is nowhere zero on $|s-s_0|<\frac{r}{2}$. On $|s-s_0|=r$ we have

$$|s-\rho|\geq\frac{r}{2}\geq|s_0-\rho|$$

for $\rho\in Z$ and

$$\left|\frac{g(s)}{g(s_0)}\right|=\left|\frac{f(s)}{f(s_0)}\prod_{\rho\in Z}\left(\frac{s_0-\rho}{s-\rho}\right)\right|\leq\left|\frac{f(s)}{f(s_0)}\right|<e^M,$$

which must hold also on $|s-s_0|\leq r$. Let

$$h(s)=\log\left(\frac{g(s)}{g(s_0)}\right)$$

on $|s-s_0|\leq\frac{r}{2}$ which is holomorphic and vanishes at s_0. Since $\operatorname{Re}h<M$ on $|s-s_0|\leq\frac{r}{2}$, it follows from the theorem of Borel-Caratheodory (with r and R replaced respectively by $\frac{3r}{8}$ and $\frac{r}{2}$) that

$$|h(s)|\leq\frac{2\,\frac{3r}{8}}{\frac{r}{2}-\frac{3r}{8}}\,M=6M$$

on $|s-s_0|\leq\frac{3r}{8}$. For $|s-s_0|\leq\frac{r}{4}$,

$$|h'(s)|=\left|\frac{1}{2\pi i}\int_{|s-s_0|=\frac{3r}{8}}\frac{h(z)}{(z-s)^2}\,dz\right|<\frac{48M}{r}.$$

Q.E.D.

Approximate Formula for Logarithmic Derivative of Riemann Zeta Function. If $\rho = \beta + i\gamma$ runs through the zeroes of $\zeta(s)$, then

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} + O\left(\log|t|\right),$$

uniformly for $-1 \leq \sigma \leq 2$, where $s = \sigma + it$. Moreover, for any given $C > 0$ the number of roots ρ of $\zeta(s)$ with $0 \leq \operatorname{Re}\rho \leq 1$ and $|\operatorname{Im}\rho - T| \leq C$ is no more than $O(\log T)$.

For the proof of the approximate formula for the logarithmic derivative of the Riemann zeta function, we need the following two theorems which were proved earlier.

Theorem on Growth Rate of Riemann Zeta Function on Vertical Lines. Let

$$\mu(\sigma) = \inf\left\{ a \in \mathbf{R} \;\middle|\; \zeta(\sigma + it) = O\left(|t|^a\right)\right\}.$$

Then

$$\mu(\sigma) = 0 \text{ if } \sigma > 1,$$
$$\mu(\sigma) = \frac{1}{2} - \sigma \text{ if } \sigma < 0.$$

Theorem of Phragmén-Lindelöf. Let $s = \sigma + it$ and $\phi(s)$ be holomorphic and of the order $O\left(e^{\epsilon|t|}\right)$ for $\epsilon > 0$ and $\sigma_1 \leq \sigma \leq \sigma_2$. Assume that $\phi(\sigma_1 + it) = O\left(|t|^{k_1}\right)$ and $\phi(\sigma_2 + it) = O\left(|t|^{k_2}\right)$. Let $k(\sigma)$ be a linear function of σ such that $k(\sigma_1) = k_1$ and $k(\sigma_2) = k_2$. Then $\phi(\sigma + it) = O\left(|t|^{k(\sigma)}\right)$.

Proof of the Approximate Formula for Logarithmic Derivative of Riemann Zeta Function. By the theorem on the growth rate of Riemann zeta function in terms of the imaginary part of its variable and by the theorem of Phragmén-Lindelöf, we have

$$|\zeta(s)| = O\left(|t|^A\right).$$

Here and below we use A as the generic symbol for a positive constant.

We apply Landau's lemma above to the case $f(s) = \zeta(s)$, $s_0 = 2 + iT$, $r = 12$, and $M = A \log T$. We use

$$\left| \frac{\zeta(s)}{\zeta(s_0)} \right| \leq T^A e^M = e^{A \log T}$$

for $|s - s_0| \leq 12$ and get

$$(\natural) \qquad \frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho - s_0| \leq 6} \frac{1}{s - \rho} + O\left(\log T\right),$$

for $|s - s_0| \leq 3$ and, in particular, for $-1 \leq \sigma \leq 2$ and $t = T$. It is clear that $|t - \gamma| \leq 1$ is a consequence of $-1 \leq \sigma \leq 2$ and $|\rho - s_0| \leq 6$. We have to worry the error term

$$(\natural\natural) \qquad \sum_{\substack{|t-\gamma|>1, \\ |\rho - s_0| \leq 6}} \frac{1}{s - \rho}.$$

Set $s = s_0$ in $(\natural)$. Then

$$\sum_{|\rho - s_0| \leq 6} \frac{1}{s_0 - \rho} = O\left(\log T\right).$$

For any root ρ of $\zeta(s)$ in $0 \leq \operatorname{Re} \rho \leq 1$, if $|\operatorname{Im} \rho - T| \leq 4$, then $1 \leq |\rho - s_0| \leq 6$. From $(\natural\natural)$ it follows that the number of roots ρ with $0 \leq \operatorname{Re} \rho \leq 1$ and $|\operatorname{Im} \rho - T| \leq 4$ is no more than $O\left(\log T\right)$. This means that for any given $C > 0$ the number of roots ρ with $0 \leq \operatorname{Re} \rho \leq 1$ and $|\operatorname{Im} \rho - T| \leq C$ is no more than $O\left(\log T\right)$. This implies that

$$\sum_{\substack{|t-\gamma|>1, \\ |\rho - s_0| \leq 6}} \frac{1}{s - \rho} = O\left(\log T\right).$$

This together with $(\natural)$ yields

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t - \gamma| \leq 1} \frac{1}{s - \rho} + O\left(\log T\right),$$

Q.E.D.