CONSTRUCTION OF MEROMORPHIC AND HOLOMORPHIC FORMS, PERIOD MATRICES, AND RIEMANN RELATIONS

§1. *Construction of Meromorphic Forms*

As a preliminary step towards the proofs of the theorems of Riemann-Roch and Abel, we are going to construct meromorphic forms on compact Riemann surfaces and then eliminate their poles by using the so-called "necklace trick" to construct holomorphic forms.

Recall that earlier we constructed a harmonic function u with a singularity on a Riemann surface M and the singularity is the same as that of some harmonic function q whose normal derivative on the boundary of a coordinate disk vanishes. There the singularity of the harmonic function q is the real part of a simple pole, but this extra piece of information plays no part in the proof. We are now going to introduce several class of harmonic functions q with singularities whose normal derivative on the boundary of a coordinate disk vanishes.

Take a point P in M and let ζ be a local holomorphic coordinate centered at P. For any natural integer ℓ we consider

$$g(\zeta) = \frac{1}{\zeta^\ell} + \frac{\zeta^\ell}{a^{2\ell}}$$

and let q be its real part. The normal derivative of q vanishes on the circle $|\zeta| = a$, because g is the pullback of

$$\frac{1}{\zeta} + \frac{\zeta}{b^{2\ell}}$$

by the map

$$\zeta \to \zeta^\ell$$

and the vanishing of the normal derivative of

$$\mathrm{Re}\left(\frac{1}{\zeta} + \frac{\zeta}{b^{2\ell}}\right)$$

on $|\zeta| = b^\ell$ was verified earlier. We can just as in our earlier situation get a harmonic function u on M minus the point $\zeta = 0$ so that $u - q$ is harmonic near $\zeta = 0$. However, since our M is compact and is in general not simply connected, we can get only a *multi-valued* harmonic conjugate v for u. Let

$$E_\ell = -\frac{1}{\ell}\,(u + \sqrt{-1}\,v).$$

The function E_ℓ of course is also multi-valued, but its differential dE_ℓ is a well-defined meromorphic form on M. The singularity of dE_ℓ at $\zeta = 0$ is simply $\frac{d\zeta}{\zeta^{\ell+1}}$ and that is reason why we choose to define E_ℓ with the coefficient $-\frac{1}{\ell}$. One important property of dE_ℓ is that its real part $-\frac{1}{\ell}du$ has no periods. So all its periods are purely imaginary if not zero.

We want also another differential form whose singularity is $\sqrt{-1}$ times that of dE_ℓ and whose periods are purely imaginary if not zero. Because of the requirement on the periods we cannot get it just by multiplying dE_ℓ by $\sqrt{-1}$. This time we let q be the *imaginary* part of

$$g(\zeta) = -\frac{1}{\zeta^\ell} + \frac{\zeta^\ell}{a^{2\ell}}.$$

That is, it is the real part of $-\sqrt{-1}\,g(\zeta)$. By using polar coordinates we check that the normal derivative of q on the circle $|\zeta| = a$ is zero. Analogous to E_ℓ we get a multi-valued meromorphic function F_ℓ. We normalize it with a suitable real constant so that the singularity of dF_ℓ at $\zeta = 0$ is $\sqrt{-1}\,\frac{d\zeta}{\zeta^{\ell+1}}$. Again the periods of dF_ℓ are purely imaginary if not zero. To emphasize the dependence of dE_ℓ and dF_ℓ on P we denote them by $dE_\ell(P)$ and $dF_\ell(P)$.

Now we take two points P and Q of M. First we assume that P is the center of a coordinate disk $\{|\zeta| = a\}$ and $|\zeta(Q)| < a$. Let $\zeta_2 = \zeta(Q)$. The reflection of Q with respect to the circle $|\zeta| = a$ is

$$\zeta_3 = \frac{a^2}{\overline{\zeta_2}}.$$

Let $g(\zeta)$ be the *multivalued* holomorphic function

$$\log \frac{\zeta}{(\zeta - \zeta_2)(\zeta - \zeta_3)}$$

on the topological closure $\overline{D_0}$ of the disk D_0 of radius a centered at P. Let $q(\zeta)$ be the real part

$$\log \left| \frac{\zeta}{(\zeta - \zeta_2)(\zeta - \zeta_3)} \right|$$

of $g(\zeta)$ which is *single-valued*.

We claim that the normal derivative of $q(\zeta)$ on the circle $|\zeta| = a$ vanishes. The logarithm and the argument of a local holomorphic coordinate can serve as the real and imaginary part of another local holomorphic coordinate. Let $\zeta = re^{i\varphi}$. By the equations of Cauchy-Riemann the vanishing of the normal derivative of q is the same as the vanishing of the derivative of the argument of

$$\frac{\zeta}{(\zeta - \zeta_2)(\zeta - \zeta_3)}$$

with respect to φ. Now using

$$\zeta_3 = \frac{\zeta\bar{\zeta}}{\bar{\zeta_2}}$$

on $|\zeta| = a$, we have

$$\frac{\frac{\bar{\zeta}}{(\bar{\zeta}-\bar{\zeta}_2)(\bar{\zeta}-\bar{\zeta}_3)}}{\frac{\zeta}{(\zeta-\zeta_2)(\zeta-\zeta_3)}} = \frac{\frac{1}{(\bar{\zeta}-\bar{\zeta}_2)\left(1-\frac{\zeta}{\zeta_2}\right)}}{\frac{1}{(\zeta-\zeta_2)\left(1-\frac{\bar{\zeta}}{\bar{\zeta}_2}\right)}} = \frac{\zeta_2}{\bar{\zeta}_2}$$

which is independent of φ. Hence the derivative of the argument of

$$\frac{\zeta}{(\zeta - \zeta_2)(\zeta - \zeta_3)}$$

with respect to φ vanishes on the circle $|\zeta| = a$ and the normal derivative of q vanishes on the circle $|\zeta| = a$.

We can construct a harmonic function u on M minus the points P and Q so that $u - q$ is harmonic near P and Q. Again we construct a multi-valued harmonic conjugate v for u and we denote by E the multi-valued function $u + iv$. Then dE is a single-valued meromorphic form on M which have simple poles at P and Q with residue 1 and -1 respectively. To emphasize the dependence of dE on P and Q we denote dE by $dE(P, Q)$. The periods of $dE(P, Q)$ are purely imaginary if not zero.

When we have two arbitrary distinct points P and Q on M we select a sequence of points

$$P = P_0, P_1, \cdots, P_n = Q$$

so that we can construct $dE(P_k, P_{k+1})$ for $0 \leq k < n$ and then we form $dE(P,Q)$ as the sum of $dE(P_k, P_{k+1})$ for $0 \leq k < n$. Note that the singularity of a meromorphic form together with its periods being purely imaginary determines the form completely, because, when we take the difference of the two forms and integrate its real part, we can a contradiction with the maximum principle of harmonic functions.

To construct nontrivial holomorphic forms, we will need a meromorphic form with residues $\sqrt{-1}$ and $-\sqrt{-1}$ at two distinct points and *almost* purely imaginary periods ("almost" in the sense that the periods are purely imaginary except for loops intersecting the branch-cut of a certain function joining the two distinct points). The existence of some real periods is essential to guaranteeing that the holomorphic forms we get are nontrivial. Actually it is not possible to imitate the preceding argument to get a meromorphic form with residues $\sqrt{-1}$ and $-\sqrt{-1}$ at two distinct points and only purely imaginary periods.

The reason is the following. To get such a meromorphic form using the preceding argument, we would have to use the *imaginary* part of some function similar to the function

$$\log \frac{\zeta}{(\zeta - \zeta_2)(\zeta - \zeta_3)}.$$

Now the real part of the function

$$\log \frac{\zeta}{(\zeta - \zeta_2)(\zeta - \zeta_3)}$$

is a single-valued function, but its imaginary part is not. The reason why the periods are only purely imaginary is that u itself is single-valued. Now if we use the imaginary part of some function similar to the function

$$\log \frac{\zeta}{(\zeta - \zeta_2)(\zeta - \zeta_3)}$$

we would only get the single-valuedness of u on $\overline{D_1}$ (where D_1 is a disk with center P which contains D_0 as a relative compact subset) after we take a cut from ζ and ζ_2. So we have to modify our statement accordingly and say that the periods of all loops avoiding the cut from ζ to ζ_2 are purely imaginary. Instead of using residues $\sqrt{-1}$ and $-\sqrt{-1}$ we want to stick with 1 and -1

and instead change the requirement on the periods so that the periods of all loops avoiding the cut from ζ to ζ_2 are *real*. More precisely we use the function

$$g(\zeta) = \log \frac{\zeta(\zeta-\zeta_3)}{(\zeta-\zeta_2)}.$$

Let L be the cut going from P to Q. Let q be the imaginary part of g. The function q can be chosen to be single-valued outside L. It has zero normal derivative on the circle $|\zeta| = a$. The verification is analogous to the previous case. Here we use

$$\frac{\zeta(\zeta-\zeta_3)}{(\zeta-\zeta_2)} \frac{\bar{\zeta}\left(\bar{\zeta}-\bar{\zeta}_3\right)}{(\bar{\zeta}-\bar{\zeta}_2)} = \zeta_3\bar{\zeta}_3.$$

Another way to verify the vanishing of the normal derivative of q at the boundary is to use the Cauchy-Riemann equations and verify that the real part of g is constant on the boundary, which follows from

$$\log\left|\frac{\zeta(\zeta-\zeta_3)}{\zeta-\zeta_2}\right| = \log|a| + \log\left|\frac{a^2-\bar{\zeta}_2\zeta}{\zeta-\zeta_2}\right| - \log(|\bar{\zeta}_2|)$$

and the fact that

$$\frac{\zeta-\zeta_2}{a^2-\bar{\zeta}_2\zeta}$$

is a Möbius transformation mapping the disk of radius a centered at the origin to itself.

By using this function q we get again a harmonic function u on $M - D_0$ so that $u - q$ is harmonic on D_1. Note that u is defined only on $M - L$. We get the multi-valued harmonic conjugate v of u and the multi-valued holomorphic function $F = \sqrt{-1}\,(u + \sqrt{-1}\,v)$. Then dF is a meromorphic form having simple poles at P and Q with residues 1 and -1 respectively and the periods of all the loops avoiding the cut L are real. The imaginary part of the period of any loop crossing L exactly once from right to left is 2π. This is because if we try to define u on $M-P-Q$ the value of u increases by 2π when one circles the point P once in the counterclockwise sense and any loop γ can be decomposed as the sum of a loop avoiding L and a positive circle around P counted as the *net* number of times as γ crosses L from right to left. To emphasize the dependence of dF on P and Q we denote it by $dF(P, Q)$.

§2. *Abelian Differentials*

We are now ready to introduce the *necklace trick* to construct holomorphic forms on any compact Riemann surface. Another name for holomorphic forms is abelian differentials of the first kind. First we have a look at the topology of a compact Riemann surface of genus g (*i.e.,* with g holes). Pick a point P_0 on the top surface of the Riemann surface M. For every hole we can issue two loops from P_0 which end at P_0. One circles the hole on the top without going into it and another one goes into the hole and comes back up again from under the handle. If one cuts up the surface along these $2g$ loops, one gets a polygon with $4g$ sides (because for each loop there is a right side and a left side). When one walks along the Riemann surface always just to the left of a loop, then one goes first along the first loop of a hole and then the second loop of the same hole and then the first loop in the reverse direction and then the second loop in the reverse direction. Then one gets to the first loop of the second hole and repeats the same pattern. This describes the way the sides of the polygon are to identified to reconstitute the Riemann surface. For every hole the pair of loops associated to it are called a pair of *conjugate crosscuts*. We denote such a pair by A and B. Such a pair has intersection number one. Two loops from different pairs have zero intersection number.

We choose a sequence of points $P_0, \cdots, P_n$ on A such that B crosses A between P_0 and P_1 once from right to left. Let $P_{n+1} = P_0$. We can now form the meromorphic forms $dF(P_k, P_{k+1})$ for $0 \leq k \leq n$. The sum $\sum_{k=0}^{n} dF(P_k, P_{k+1})$ is a holomorphic form on M because of the cancellation of the singularities of $dF(P_{k-1}, P_k)$ and $dF(P_k, P_{k+1})$ at P_k. The real part of the period of $\sum_{k=0}^{n} dF(P_k, P_{k+1})$ along B is 2π and the period of $\sum_{k=0}^{n} dF(P_k, P_{k+1})$ along any loop with zero intersection number with A is purely imaginary. We can normalize this form by a real constant so that its period along B has real part 1 and its period along any loop with zero intersection number with A is purely imaginary. By interchanging the roles of A and B we get another holomorphic form.

We can now conclude that there are precisely g $\mathbf{C}$-linearly independent holomorphic forms on M. We have g pairs of conjugate crosscuts A_k, B_k $(1 \leq k \leq g)$. For each pair we have two holomorphic forms α_k and β_k so that the real part of the period of α_k (respectively β_k) along A_k (respectively B_k) is 1 and the period of α_k (respectively β_k) along the other $2g-1$ loops are purely imaginary. So the $2g$ holomorphic forms are $\mathbf{R}$-linearly independent. Since the set of all holomorphic forms forms a $\mathbf{C}$-vector space, it follows that there are g $\mathbf{C}$-linearly independent holomorphic forms on the Riemann surface.

We saw earlier that by making the real part of all periods vanish and using the maximum principle for harmonic functions we know that there cannot be more than g $\mathbf{C}$-linearly independent holomorphic forms on the Riemann surface. Thus we have precisely g $\mathbf{C}$-linearly independent holomorphic forms on the Riemann surface.

§3. *Riemann relations for the periods*

When we discuss elliptic functions we get three fundamental properties of the zeroes and poles of an elliptic function by applying the Cauchy theorem and the residue theorem to a fundamental parallelogram and using cancellation of integrals along corresponding sides. The Cauchy theorem and the residue theorem are both consequences of applying Stokes' theorem. Now for a general Riemann surface we have a fundamental polygon with $4g$ sides instead of a fundamental parallelogram. We can also apply Stokes' theorem to the fundamental polygon and get some conclusions. These are the Riemann relations. One is an identity and one is an inequality.

Take two holomorphic forms α and β on the Riemann surface. On the fundamental polygon they are the total differentials of two holomorphic functions φ and ψ. We have conjugate crosscuts A_k, B_k $(1 \leq k \leq g)$ so that the sides of the fundamental polygon in the counter-clockwise sense are

$$A_1, B_1, -A_1, -B_1, \cdots, A_g, B_g, -A_g, -B_g.$$

We denote the period of $d\varphi$ along A_k by $\varphi(A_k)$ which is equal to the difference of the value of φ at the two end-points of A_k. The notations $\varphi(B_k)$, $\psi(A_k)$, $\psi(B_k)$ have similar meanings. Let Ω denote the fundamental polygon. By Cauchy's theorem

$$0 = \int_{\partial\Omega} \varphi d\psi = \sum_{k=1}^{g} \left(\int_{A_k} \{\varphi - (\varphi + \varphi(B_k))\} \, d\psi + \int_{B_k} \{\varphi - (\varphi - \varphi(A_k))\} \, d\psi \right)$$

$$= \sum_{k=1}^{g} \left(\varphi(A_k) \int_{B_k} d\psi - \varphi(B_k) \int_{A_k} d\psi \right) = \sum_{k=1}^{g} \left(\varphi(A_k)\psi(B_k) - \varphi(B_k)\psi(A_k) \right).$$

The first Riemann relation is

$$\sum_{k=1}^{g} \left(\varphi(A_k)\psi(B_k) - \varphi(B_k)\psi(A_k) \right) = 0$$

for any abelian differentials of first kind $d\varphi$ and $d\psi$.

The second Riemann relation is obtained by using

$$\frac{1}{2\sqrt{-1}}\int_{\partial\Omega}\bar{\varphi}d\varphi=\frac{1}{2\sqrt{-1}}\int_{\Omega}d\bar{\varphi}\wedge d\varphi>0.$$

This is true, because at points where $d\varphi$ is nonzero we can use φ as a local holomorphic coordinate and for any local holomorphic coordinate $z=x+\sqrt{-1}\,y$ one has

$$\frac{1}{2\sqrt{-1}}\,d\bar{z}\wedge dz=dx\wedge dy.$$

The boundary integral

$$\frac{1}{2\sqrt{-1}}\int_{\partial\Omega}\bar{\varphi}d\varphi$$

can be decomposed as in the first Riemann relation to give

$$\frac{1}{2\sqrt{-1}}\int_{\partial\Omega}\bar{\varphi}d\varphi=\frac{1}{2\sqrt{-1}}\sum_{k=1}^{g}\left(\bar{\varphi}(A_k)\varphi(B_k)-\bar{\varphi}(B_k)\varphi(A_k)\right).$$

The second Riemann relation says that

$$\frac{1}{2\sqrt{-1}}\sum_{k=1}^{g}\left(\bar{\varphi}(A_k)\varphi(B_k)-\bar{\varphi}(B_k)\varphi(A_k)\right)>0.$$

We can interpret these two relations in terms of the deRham isomorphism. The deRham isomorphism establishes a one-one correspondence between the group of closed p-forms modulo exact p-forms and the p-th cohomology group with coefficients in $\mathbf{C}$. The correspondence is defined by integration over p-cycles and is compatible with exterior and cup products. First we have the intersection matrix J whose entries are the intersection numbers of the basis

$$A_1,\cdots,A_g,B_1,\cdots,B_g.$$

We choose the orientations of A_k, B_k so that the intersection number of A_k and B_k is 1 and the intersection number of B_k and A_k is -1. Then J is the $2g\times 2g$ skew-symmetric matrix

$$\begin{pmatrix}0 & E_g\\ -E_g & 0\end{pmatrix}$$

where E_g is the $p \times p$ identity matrix. The holomorphic differential form α is closed, because locally it is of the form $\alpha = f(z)dz$ for some holomorphic function f and

$$d\alpha = \frac{d}{dz} f(z)\, dz \wedge dz = 0.$$

So α represents a cohomology class which is an element $[\alpha]$ of $H^1(M, \mathbf{C})$. Now $H_1(M, \mathbf{C})$ is dual to $H^1(M, \mathbf{C})$ and a basis of $H_1(M, \mathbf{C})$ is

$$A_1, \cdots, A_g, B_1, \cdots, B_g.$$

So with respect to this basis of $H_1(M, \mathbf{C})$ the cohomology class $[\alpha]$ is represented by the column $2g$-vector $[\varphi]$ whose entries are

$$\varphi(A_\nu) = \int_{A_\nu} \alpha, \qquad \varphi(B_\nu) = \int_{B_\nu} \alpha \qquad (1 \leq \nu \leq g).$$

The 2-form $\alpha \wedge \beta$ represents a cohomology class $[\alpha \wedge \beta]$ which is an element of $H^2(M, \mathbf{C})$. With respect to the homology class $H_2(M, \mathbf{C})$ represented by M the cohomology class $[\alpha \wedge \beta]$ is given by the number

$$\int_M \alpha \wedge \beta.$$

By the isomorphism of deRham the cohomology class $[\alpha \wedge \beta]$ is equal to the cohomology class represented by the cup product of the two cohomology classes $[\alpha]$ and $[\beta]$. This cup product is represented with respect to the basis M of $H_2(M, \mathbf{C})$ by $[\varphi]' J [\varphi]$, where $[\varphi]'$ means the transpose of the column vector $[\varphi]$, because the cup product (with respect to the basis M of $H_2(M, \mathbf{C})$) of two cohomology classes given by two loops is simply the intersection number of the two loops. The exterior product $\alpha \wedge \beta$ is identically zero, because locally $\alpha = f(z)dz$ and $\beta = g(z)dz$ for some holomorphic functions $f(z)$ and $g(z)$. Thus

$$[\varphi]' J [\varphi] = 0$$

which is the first Riemann relation

$$\sum_{k=1}^{g} \left(\varphi(A_k)\psi(B_k) - \varphi(B_k)\psi(A_k) \right) = 0.$$

For the second Riemann relation instead of β we use the complex conjugate $\bar{\alpha}$ of α in our use of the deRham isomorphism. Then

$$\int_M \bar{\alpha} \wedge \alpha = [\bar{\varphi}]' J [\varphi].$$

Now locally

$$\frac{1}{2\sqrt{-1}} \, \bar{\alpha} \wedge \alpha = \frac{1}{2i} |f(z)|^2 d\bar{z} \wedge dz = |f(z)|^2 dx \wedge dy \geq 0$$

with equality only at points where $f(z)$ is zero. Thus we have

$$[\bar{\varphi}]' J [\varphi] > 0$$

which is simply the second Riemann relation

$$\frac{1}{2\sqrt{-1}} \sum_{k=1}^{g} \left(\bar{\varphi}(A_k) \varphi(B_k) - \bar{\varphi}(B_k) \varphi(A_k) \right) > 0.$$

§4. *Period Matrices.*

On a compact Riemann surface M of genus g we have g **C**-linearly independent holomorphic forms on M. On the fundamental polygon of $4g$ sides each of these g holomorphic forms is the differential of a holomorphic function w_ν $(1 \leq \nu \leq g)$. On the compact Riemann surface M itself the holomorphic functions w_ν are *multi-valued*. The g **C**-linearly independent holomorphic differential forms on M are $dw_1, \cdots, dw_g$. The periods of dw_ν are

$$\int_{A_\nu} dw_\nu, \quad \int_{B_\nu} dw_\nu \quad (1 \leq \nu \leq g)$$

and we denote by $[w_\nu]$ the column $2g$-vector formed by these periods. We call the $2g \times g$ matrix

$$Q = \begin{pmatrix} [w_1] & \cdots & [w_g] \end{pmatrix}$$

the *period matrix* of M for the basis $dw_1, \cdots, dw_g$ of holomorphic differential forms and the basis

$$A_1, \cdots, A_g, B_1, \cdots, B_g$$

of the first homology group. The first Riemann relation gives the matrix equation $Q'JQ = 0$, where Q' is the transpose of Q. Let $H = \frac{1}{2i}\bar{Q}'JQ$. The second Riemann relation tells us that the matrix H is positive-definite. In particular, H is Hermitian.

Let us now examine the effect of changing the bases $dw_1, \cdots, dw_g$ and

$$A_1, \cdots, A_g, B_1, \cdots, B_g.$$

For the change of the basis $dw_1, \cdots, dw_g$ the period matrix Q is multiplied on the right by a nonsingular matrix with entries in $\mathbf{C}$. For the change of the basis

$$A_1, \cdots, A_g, B_1, \cdots, B_g$$

the period matrix Q is multiplied on the left by a nonsingular matrix with entries in $\mathbf{Z}$. We can reduce Q to a simpler form by changing the bases.

Write Q in the form $\begin{pmatrix} F \\ G \end{pmatrix}$, where F and G are $g \times g$ matrices. The matrix F is nonsingular, otherwise we can find a non-identically-zero holomorphic form $d\varphi$ whose period $\varphi(A_\nu)$ over each A_ν is zero, which would contradict the second Riemann relation

$$\frac{1}{2\sqrt{-1}} \sum_{k=1}^{g} \left(\bar{\varphi}(A_k)\varphi(B_k) - \bar{\varphi}(B_k)\varphi(A_k)\right) > 0.$$

So we can choose a change of the basis $dw_1, \cdots, dw_g$ so that the new period matrix is equal to QF^{-1}. Thus Q equals $\begin{pmatrix} E \\ Z \end{pmatrix}$, where E is the $p \times p$ identity matrix and $Z = QF^{-1}$. In this case we call the basis $dw_1, \cdots, dw_g$ *a normal basis* and the period matrix is said to be *normalized*. The second component Z of the period matrix Q is also referred to as the period matrix in this case.

For a normal basis the two Riemann relations simply say that Z is symmetric and the imaginary part of Z is positive definite, because

$$Q'JQ = \begin{pmatrix} E & Z' \end{pmatrix} \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} E \\ Z \end{pmatrix} = Z - Z',$$

$$\frac{1}{2\sqrt{-1}} \bar{Q}'JQ = \frac{1}{2\sqrt{-1}} \begin{pmatrix} E & \bar{Z}' \end{pmatrix} \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} E \\ Z \end{pmatrix}$$

$$= \frac{1}{2\sqrt{-1}} (Z - \bar{Z}') = \frac{1}{2\sqrt{-1}} (Z - \bar{Z}).$$

We now examine the effect of the change of basis $A_1, \cdots, A_g, B_1, \cdots, B_g$. Let

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

denote the $2p \times 2p$ matrix with entries in $\mathbf{Z}$ which is the transformation matrix for the change of basis of the first homology group. First we want the new basis to have the same intersection matrix J. This means that we have the relation $T'JT = J$. A real $2p \times 2p$ matrix L is said to be *symplectic* if $L'JL = J$. Note that by taking the determinant of both sides of $T'JT = J$ we know that the determinant of a symplectic matrix is always ± 1. A symplectic matrix whose entries are integers is called a *modular* matrix. So we consider only modular matrices with entries in $\mathbf{Z}$. After we change the basis

$$A_1, \cdots, A_g, B_1, \cdots, B_g$$

of the first homology group we want to keep the period matrix normalized. So we make a change of the basis $dw_1, \cdots, dw_g$ after we change the basis

$$A_1, \cdots, A_g, B_1, \cdots, B_g.$$

The (second component of the) period matrix Z is transformed to

$$(C + DZ)(A + BZ)^{-1}.$$

The transformation from Z to

$$(C + DZ)(A + BZ)^{-1}$$

is known as a *modular substitution* when

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a modular matrix. The set of all symmetric complex matrices Z with positive definite imaginary parts is called the *generalized upper half-plane*. A modular substitution is the analog of a Möbius transformation for the case of the generalized upper half-plane.