Darboux's Proof of the Addition Formula for Elliptic Functions

In his 1867 paper referenced below, Darboux gave a proof of the addition formula for elliptic functions which is based on the conservation of a momentum-type expression for a physical system consisting of two pendulums moving in a synchronized fashion.

Jean Gaston Darboux, Ann. de l'École Norm., ***IV****, p.85 (1867).*

Consider the two elliptic functions $\xi(u) = \operatorname{sn} u$ and $\eta(v) = \operatorname{sn} v$ given by the differential equations by

$$\frac{d\xi}{du} = \sqrt{F(\xi)}$$

and

$$\frac{d\eta}{dv} = \sqrt{F(\eta)}$$

with $\xi(0) = 0$ and $\eta(0) = 0$, where

$$F(x) = (1 - x^2)(1 - k^2x^2).$$

Assume that $u + v = c$, where c is a constant. Then

$$\frac{d\eta}{du} = -\sqrt{F(\eta)}.$$

Differentiating

$$\frac{d\xi}{du} = \sqrt{F(\xi)}$$

once yields

$$\frac{d^2\xi}{du^2} = \frac{1}{2}F'(\xi).$$

Likewise

$$\frac{d^2\eta}{du^2} = \frac{1}{2}F'(\eta).$$

We now introduce the following physical system of two pendulums. The formula of motion for a pendulum of length a and initial angle α is given by

$$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\operatorname{sn}(\sqrt{\frac{g}{a}}t),$$

where θ is the angle made with the vertical line. We normalize the constants so that $g = a$ and we have

$$\frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}} = \operatorname{sn} t.$$

We consider the motion of two pendulums with angles θ and φ respectively made with the vertical line. Let

$$\xi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}$$

and

$$\eta = \frac{\sin\frac{\varphi}{2}}{\sin\frac{\alpha}{2}}.$$

We assume also the times for the motion of each angle have a phase difference so that the time for ξ is u and the time for η is v with $u+v=c$. This means that when the time u goes in the positive direction, the time v goes in the negative direction. To make both go in the positive direction, we use time $-v$ for the motion for the angel φ and change the sign for the definition of φ. So we have the equations of motion

$$\xi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}} = \operatorname{sn} u$$

and

$$\eta = \frac{\sin\frac{\varphi}{2}}{\sin\frac{\alpha}{2}} = -\operatorname{sn}(-v).$$

Thus the phase difference in time is that the time u is for ξ and the time $-v$ is for η and $u - (-v) = c$. We consider the motion of a particle in the (ξ, η)-plane whose coordinates move according to the functions $\xi = \xi(u)$ and $\eta = \eta(-v)$ from the two pendulums with the phase constraint $u + v = c$. In the case of the sine function the addition formula involves

$$\sin\frac{\theta}{2}\left(\sin\frac{\varphi}{2}\right)' + \sin\frac{\varphi}{2}\left(\sin\frac{\theta}{2}\right)'.$$

Since the time variables u and v now go in opposite directions, we should consider the expression

$$\eta\left(\frac{d\xi}{du}\right) - \xi\left(\frac{d\eta}{du}\right).$$

This expression is some sort of angular momentum. To get a conservation law involving

$$\eta\left(\frac{d\xi}{du}\right)-\xi\left(\frac{d\eta}{du}\right),$$

we should compute

$$\eta\frac{d^2\xi}{du^2}-\xi\frac{d^2\eta}{du^2}$$

which is the torque for the particle at (ξ,η) on the (ξ,η)-plane with respect to the origin. Then the conservation law for the "angular momentum" is the first integral of motion. The mathematical procedure is as follows.

$$\eta\frac{d^2\xi}{du^2}-\xi\frac{d^2\eta}{du^2}=2\ k^2\xi\eta(\xi^2-\eta^2),$$

$$\eta^2\left(\frac{d\xi}{du}\right)^2-\xi^2\left(\frac{d\eta}{du}\right)^2=\eta^2-\xi^2+k^2\xi^2\eta^2\left(\xi^2-\eta^2\right).$$

Taking the quotients of the above two equations and factoring

$$\eta^2\left(\frac{d\xi}{du}\right)^2-\xi^2\left(\frac{d\eta}{du}\right)^2,$$

we get

$$\left(\eta\frac{d^2\xi}{du^2}-\xi\frac{d^2\eta}{du^2}\right)\left(\eta\left(\frac{d\xi}{du}\right)-\xi\left(\frac{d\eta}{du}\right)\right)^{-1}$$
$$=2k^2\xi\eta\left(\eta\left(\frac{d\xi}{du}\right)+\xi\left(\frac{d\eta}{du}\right)\right)\left(k^2\xi^2\eta^2-1\right)^{-1}.$$

We can integrating this and get

$$\log\left(\eta\left(\frac{d\xi}{du}\right)-\xi\left(\frac{d\eta}{du}\right)\right)=\log\left(k^2\xi^2\eta^2-1\right)+\text{constant}.$$

In other words, we have

$$\left(\eta\left(\frac{d\xi}{du}\right)-\xi\left(\frac{d\eta}{du}\right)\right)\left(1-k^2\xi^2\eta^2\right)^{-1}=C.$$

Plucking in the values for $\frac{d\xi}{du}$ and $\frac{d\eta}{du}$, we finally obtain

$$(\dagger) \qquad \left(\eta\sqrt{F(\xi)} + \xi\sqrt{F(\eta)}\right)\left(1 - k^2\xi^2\eta^2\right)^{-1} = C.$$

Note that u, v, ξ, and η are all related and there is only one independent variable which can be chosen to be any of the four. We more or less could interpret the above equation as some law of conversation of angular momentum. We now determine the "angular momentum" C in terms of the phase difference c. Suppose $\eta = \eta_0$ corresponds to $\xi = 0$. Then we conclude that $C = \eta_0$. Write $\xi = \operatorname{sn} u$ and $\eta = \operatorname{sn} v$. When $\xi = 0$, we have $u = 0$ and $v = c$ and $\eta_0 = \operatorname{sn} c$. Hence $C = \operatorname{sn} c$ or the "angular momentum" is equal to sn of the phase difference c. This relation († is the same as the addition formula. To interpret (†) as the addition formula, we write $C = \operatorname{sn}(u+v)$ and the equation (†) becomes

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$