

Dirichlet Series, Abscissas of Convergence, and Growth Rate Along Vertical Lines

Radius of Convergence for Power Series and Abscissa of Convergence for Dirichlet Series. The abscissa of convergence (or absolute convergence) for a Dirichlet series is the analog of the radius convergence for a power series. One important difference is the gap between the abscissa of convergence and the abscissa of absolute convergence. The gap can achieve the maximum value of 1. The interesting case for us is

$$\left(1 - \frac{1}{2^{s-1}}\right) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

whose abscissa of convergence is $\operatorname{Re} s = 0$ and whose abscissa of absolute convergence is $\operatorname{Re} s = 1$. We now derive these properties of the abscissa of convergence and the abscissa of absolute convergence.

Uniform Convergence on Sector (Use of Summation by Parts to Determine Abscissas of Convergence). Let

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series. If the series converges at some s_0 , then for any $0 < \delta < \frac{\pi}{2}$ it converges uniformly on

$$-\frac{\pi}{2} + \delta \leq \arg(s - s_0) \leq \frac{\pi}{2} - \delta.$$

For the verification it suffices to consider the case $s_0 = 0$. Let $r_n = \sum_{k=n+1}^{\infty} a_k$. By summation by parts

$$\sum_{n=M}^N \frac{a_n}{n^s} = \sum_{n=M}^N \frac{r_{n-1} - r_n}{n^s} = \sum_{n=M}^N r_n \left(\frac{1}{(n+1)^s} - \frac{1}{n^s} \right) + \frac{r_{M-1}}{M^s} - \frac{r_N}{(N+1)^s}.$$

Since

$$\left| \frac{1}{(n+1)^s} - \frac{1}{n^s} \right| = \left| s \int_{u=n}^{n+1} \frac{du}{u^{s+1}} \right| \leq |s| \int_{u=n}^{n+1} \frac{du}{u^{\sigma+1}} = \frac{|s|}{\sigma} \left(\frac{1}{(n+1)^{\sigma}} - \frac{1}{n^{\sigma}} \right),$$

it follows that, when $|r_n| < \varepsilon$ for $M \geq n_0$,

$$\begin{aligned} \left| \sum_{n=M}^N \frac{a_n}{n^s} \right| &< \frac{\varepsilon|s|}{\sigma} \sum_{n=M}^N \left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right) + \frac{\varepsilon}{M^\sigma} + \frac{\varepsilon}{(N+1)^\sigma} \\ &= \frac{\varepsilon|s|}{\sigma} \left(\frac{1}{M^\sigma} - \frac{1}{(N+1)^\sigma} \right) + \frac{\varepsilon}{M^\sigma} + \frac{\varepsilon}{(N+1)^\sigma} \\ &< \frac{2\varepsilon|s|}{\sigma} + 2\varepsilon \leq 2\varepsilon \left(\frac{1}{\cos \delta} + 1 \right). \end{aligned}$$

Definition of Abscissa of Convergence. The infimum σ_0 of all $\operatorname{Re} s$ such that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is convergent is the *abscissa of convergence* of the Dirichlet series.

Definition of Abscissa of Absolute Convergence. The abscissa of convergence of the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^s}$$

is the *abscissa of absolute convergence* $\bar{\sigma}$ of the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Gap between Abscissa of Convergence and Abscissa of Absolute Convergence. The relation between σ_0 and $\bar{\sigma}$ is $\sigma_0 \leq \bar{\sigma} \leq \sigma_0 + 1$, because the convergence of

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

implies

$$\sup_{1 \leq n < \infty} \frac{|a_n|}{n^\sigma} < \infty,$$

which in turn implies that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma+1+\delta}} < \infty$$

for any $\delta > 0$. The relation σ_0 and $\bar{\sigma}$ is sharp, as is shown by the example

$$\left(1 - \frac{1}{2^{s-1}}\right) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

Growth Rate of Dirichlet Series Along Vertical Lines. Let $f(s)$ be the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

with abscissa of convergence σ_0 . Let $\varepsilon > 0$. Then

$$f(s) = O(|t|^{1-(\sigma-\sigma_0)+\varepsilon})$$

uniformly on $\sigma_0 + \delta \leq \sigma \leq \sigma_0 + 1$ for any $\delta > 0$, where $s = \sigma + it$. The proof is by summation by parts and by cutting off the partial sum comparable to the imaginary part of the variable. By a translation along the real part of the variable we can assume without loss of generality that σ_0 is negative and very close to 0 (relative to ε) so that $\sum_{n=1}^{\infty} a_n$ converges and we need only verify

$$f(s) = O(|t|^{1-\sigma+\varepsilon})$$

uniformly for $\delta \leq \sigma \leq 1$. Let $s_n = \sum_{k=1}^n a_k$. Summation by parts from $n = M + 1$ to $n = N$ yields

$$\sum_{n=1}^N \frac{a_n}{n^s} = \sum_{n=1}^M \frac{a_n}{n^s} + \sum_{n=M+1}^N s_n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) - \frac{s_M}{(M+1)^s} + \frac{s_N}{(N+1)^s}.$$

It follows from $\sigma > 0$ that as $N \rightarrow \infty$ we have

$$f(s) = \sum_{n=1}^M \frac{a_n}{n^s} + \sum_{n=M+1}^{\infty} s_n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) - \frac{s_M}{(M+1)^s}.$$

Let A be a bound for $|a_n|$ and $|s_n|$. Since

$$\left| \frac{1}{(n+1)^s} - \frac{1}{n^s} \right| = \left| s \int_{u=n}^{n+1} \frac{du}{u^{s+1}} \right| \leq |s| \int_{u=n}^{n+1} \frac{du}{u^{\sigma+1}} = \frac{|s|}{\sigma} \left(\frac{1}{(n+1)^\sigma} - \frac{1}{n^\sigma} \right),$$

it follows that

$$|f(s)| \leq A \left(\sum_{n=1}^M \frac{1}{n^\sigma} + \sum_{n=M+1}^{\infty} \frac{|s|}{\sigma} \left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right) + \frac{1}{(M+1)^\sigma} \right).$$

Since

$$\sum_{n=2}^M \frac{1}{n^\sigma} \leq \int_{x=1}^M \frac{dx}{x^\sigma} = \frac{1}{1-\sigma} (M^{1-\sigma} - 1),$$

it follows that for $0 < \sigma < 1$ and $t > 0$,

$$|f(s)| \leq A \left(\frac{1}{1-\sigma} M^{1-\sigma} + t \frac{1}{M^\sigma} + \frac{1}{(M+1)^\sigma} \right).$$

We obtain

$$f(s) = O(t^{1-\sigma})$$

by choosing M to be the integral part of t .

Growth Rate of Riemann Zeta Function on the Vertical Strip $0 \leq \operatorname{Re} z \leq 1$.

Applying the result to

$$\left(1 - \frac{1}{2^{s-1}}\right) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

whose abscissa of convergence $\sigma_0 = 0$ and whose abscissa of absolute convergence $\bar{\sigma} = 1$, we get

$$\zeta(s) = O(|t|^{1-\sigma+\varepsilon}) \quad \text{for } 0 < \sigma \leq 1 \text{ and } \varepsilon > 0.$$