Explicit Formula for Logarithmic Derivative of Riemann Zeta Function

The explicit formula for the logarithmic derivative of the Riemann zeta function is the application to it of the Perron formula with error estimates. The goal is to express $\sum_{n<x}\Lambda(n)$ in terms of x with error estimates which depends on an adjustable parameter T coming from applying the Perron formula to a vertical line with total length $2T$ instead of to an infinite vertical line which corresponds to setting $T=\infty$. The estimate for $\sum_{n<x}\Lambda(n)$ will be used later to relate the containment of the zero-set of $\zeta(s)$ in a vertical strip to the order of the error term of $-x+\sum_{n<x}\Lambda(n)$. It will be applied also later to give a proof of the Prime Number Theorem. Since we apply the Perron formula only to a vertical line with total length $2T$ and keep the error term as a function of T instead of passing immediately to the limit $T\to\infty$, we will need the Perron formula with error estimates. We will do it first for a single term in the form of the following lemma.

Perron Lemma with Error Estimate. For $c>0$ and $T>0$,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} x^s\,\frac{ds}{s}=\begin{cases} E(x)+O\left(\frac{x^c}{T|\log x|}\right) & \text{if} \quad x\neq 1 \\ \\ E(x)+O\left(\frac{1}{T}\right) & \text{if} \quad x=1\,, \end{cases}$$

where

$$E(x)=\begin{cases} 0 & \text{if} \quad 0<x<1 \\ \\ \frac{1}{2} & \text{if} \quad x=1 \\ \\ 1 & \text{if} \quad x>1\,. \end{cases}$$

Proof. We treat the three cases: (i) $x=1$, (ii) $x>1$, and (iii) $x<1$ separately.

Case (i). Suppose $x=1$. Consider the branch $\log z$ of log so that the angle θ with $z=re^{i\theta}$ is confined to $-\pi<\theta<\pi$ and $\log z=\log r+i\theta$. Then

$$\int_{c-iT}^{c+iT}\frac{ds}{s}=\log\left(c+iT\right)-\log\left(c-iT\right).$$

From the choice of the branch of the logarithmic function,

$$\operatorname{Re}\left(\log\left(c+iT\right)-\log\left(c-iT\right)\right)=\log\left|c+iT\right|-\log\left|c-iT\right|=0$$

and

$$\operatorname{Im}\left(\log\left(c+iT\right)-\log\left(c-iT\right)\right)=\arctan\left(\frac{T}{c}\right)-\arctan\left(\frac{-T}{c}\right)=2\arctan\left(\frac{T}{c}\right)$$

which approaches π as $T \to \infty$. Hence

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} x^s\,\frac{ds}{s}=\frac{1}{2}$$

when $x=1$.

Case (ii). Suppose $x>1$. Choose $a<0$ and consider the contour integration of

$$\frac{1}{2\pi i}\,x^s\,\frac{ds}{s}=\frac{1}{2\pi i}\,e^{s\log x}\,\frac{ds}{s}$$

along the boundary of the rectangle with vertices at

$$a-iT,\ \ c-iT,\ \ c+iT, a+iT.$$

Since the residue of

$$\frac{x^s}{s}=\frac{e^{s\log x}}{s}$$

is 1 at its only pole $s=0$, the contour integral is equal to 1.

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{a+iT} x^s\,\frac{ds}{s}\right|\le\frac{1}{2\pi}\int_{-T}^{T} x^a\,\frac{dt}{\sqrt{a^2+t^2}}\to 0$$

as $a\to-\infty$, because $x>1$.

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{c-iT} x^s\,\frac{ds}{s}\right|\le\frac{1}{2\pi}\int_{a}^{c} x^\sigma\,\frac{d\sigma}{\sqrt{\sigma^2+T^2}},$$
$$\left|\frac{1}{2\pi i}\int_{a+iT}^{c+iT} x^s\,\frac{ds}{s}\right|\le\frac{1}{2\pi}\int_{a}^{c} x^\sigma\,\frac{d\sigma}{\sqrt{\sigma^2+T^2}},$$

where $s=\sigma+it$. Thus,

$$\left|\frac{1}{2\pi i}\int_{a-iT}^{c-iT} x^s\,\frac{ds}{s}+\frac{1}{2\pi i}\int_{a+iT}^{c+iT} x^s\,\frac{ds}{s}\right|\le\frac{\sqrt{2}}{\pi}\int_{a}^{c} x^\sigma\,\frac{d\sigma}{|\sigma|+T},$$

because $\left(|\sigma|+T\right)^2\le 2\left(\sigma^2+T^2\right)$. Finally we have

$$\int_{a}^{c} x^\sigma\,\frac{d\sigma}{|\sigma|+T}\le\int_{-\infty}^{c} x^\sigma\,\frac{d\sigma}{T}=\frac{x^c}{T\log x}.$$

Case (iii). Suppose $0 < x < 1$. Choose $b > c$ and consider the contour integration of
$$\frac{1}{2\pi i}\, x^s \,\frac{ds}{s} = \frac{1}{2\pi i}\, e^{s\log x}\,\frac{ds}{s}$$
along the boundary of the rectangle with vertices at
$$c - iT,\ b - iT,\ b + iT, c + iT.$$
Since the residue of
$$\frac{x^s}{s} = \frac{e^{s\log x}}{s}$$
is 1 at its only pole $s = 0$, the contour integral is equal to 1.
$$\left|\frac{1}{2\pi i}\int_{b-iT}^{b+iT} x^s\,\frac{ds}{s}\right| \le \frac{1}{2\pi}\int_{-T}^{T} x^b\,\frac{dt}{\sqrt{b^2+t^2}} \to 0$$
as $b \to \infty$, because $0 < x < 1$.
$$\left|\frac{1}{2\pi i}\int_{c-iT}^{b-iT} x^s\,\frac{ds}{s}\right| \le \frac{1}{2\pi}\int_{c}^{b} x^\sigma\,\frac{d\sigma}{\sqrt{\sigma^2+T^2}},$$
$$\left|\frac{1}{2\pi i}\int_{c+iT}^{b+iT} x^s\,\frac{ds}{s}\right| \le \frac{1}{2\pi}\int_{c}^{b} x^\sigma\,\frac{d\sigma}{\sqrt{\sigma^2+T^2}},$$
where $s = \sigma + it$. Thus,
$$\left|\frac{1}{2\pi i}\int_{c-iT}^{b-iT} x^s\,\frac{ds}{s} + \frac{1}{2\pi i}\int_{c+iT}^{b+iT} x^s\,\frac{ds}{s}\right| \le \frac{\sqrt{2}}{\pi}\int_{c}^{b} x^\sigma\,\frac{d\sigma}{\sigma+T},$$
because $(\sigma+T)^2 \le 2\left(\sigma^2+T^2\right)$.
$$\int_c^b x^\sigma\,\frac{d\sigma}{\sigma+T} \le \int_c^\infty x^\sigma\,\frac{d\sigma}{\sigma+T} \le \int_c^\infty x^\sigma\,\frac{d\sigma}{T} = \frac{x^c}{T\,|\log x|}.$$

Application of Perron Lemma to Logarithmic Derivative of Riemann Zeta Function. The explicit formula for the logarithmic derivative of the Riemann zeta function will be derived by applying residue calculation to the contour integration of
$$\frac{-1}{2\pi i}\,\frac{\zeta'(s)}{\zeta(s)}\, x^s\,\frac{ds}{s}$$

over the rectangle with vertices at $-M-iT,\ c-iT,\ c+iT,\ -M+iT$, for $c>1$ and x equal to a non-integer >1, when M is allowed to go to $-\infty$. There are four integrals over the four sides of the rectangle. For the three sides $[-M-iT,\,-M+iT]$, $[-M-iT,\,c-iT]$, and $[-M+iT,\,c+iT]$, we are going to use

$$\frac{-1}{2\pi i}\,\frac{\zeta'(s)}{\zeta(s)}\,x^s\,\frac{ds}{s}$$

for their evaluation. However, for the side from $c-iT$ to $c+iT$ we are going to use the formula

$$-\frac{\zeta'(s)}{\zeta(s)}=\sum_{n\in\mathbb{N}}\frac{\Lambda(n)}{n^s}$$

to transform the integral to

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\sum_{n\in\mathbb{N}}\Lambda(n)\left(\frac{x}{n}\right)^s\,\frac{ds}{s},$$

which, by the Perron lemma with error estimates, is equal to

$$\sum_{n<x}\Lambda(n)\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^s\,\frac{ds}{s}$$

$$=\sum_{n<x}\Lambda(n)+O\left(\sum_{\substack{n>1\\ n\neq x}}\Lambda(n)\left(\frac{x}{n}\right)^c\frac{1}{T\left|\log\left(\frac{x}{n}\right)\right|}\right)+O\left(\frac{\Lambda(x)}{T}\right),$$

where the value $\Lambda(x)$ is defined as 0 if x is not an integer. The term $O\left(\frac{\Lambda(x)}{T}\right)$ on the right-hand side is only for the case $x=n$.

We have to worry about the denominator $\left|\log\left(\frac{x}{n}\right)\right|$ in

$$O\left(\sum_{n\neq x}\Lambda(n)\left(\frac{x}{n}\right)^c\frac{1}{T\left|\log\left(\frac{x}{n}\right)\right|}\right)$$

Consider the set of n such that $n<\frac{2x}{3}$ or $n>\frac{3x}{2}$. Then for such n we have

$$\left|\log\left(\frac{x}{n}\right)\right|>\log\frac{3}{2}$$

and

$$O\left(\sum_{\substack{n<\frac{2x}{3}\\ \text{or}\ \ n>\frac{3x}{2}}}\Lambda(n)\left(\frac{x}{n}\right)^c\frac{1}{T}\right)=O\left(\sum_{n>1}\Lambda(n)\left(\frac{x}{n}\right)^c\frac{1}{T}\right)$$
$$=O\left(\frac{\zeta'(c)}{\zeta(c)}\,\frac{x^c}{T}\right)=O\left(\frac{1}{|c-1|}\,\frac{x^c}{T}\right),$$

because for $c>1$ we have

$$\frac{\zeta'(c)}{\zeta(c)}=O\left(\frac{1}{|c-1|}\right).$$

We now estimate

$$O\left(\sum_{\frac{2x}{3}<n<\frac{3x}{2}}\Lambda(n)\left(\frac{x}{n}\right)^c\frac{1}{T\left|\log\left(\frac{x}{n}\right)\right|}\right).$$

In the range $\frac{2x}{3}<n<\frac{3x}{2}$, we have $\frac{2}{3}\,n<x<\frac{3}{2}\,n$ and $-\frac{1}{3}\,n<x-n<\frac{1}{2}\,n$ and $|x-n|\le\frac{1}{2}\,n<\frac{4}{3}\,x$. Let $\langle x\rangle$ be the distance between x and the integer closest to x so that $\langle x\rangle\le\frac{1}{2}$. Note that for $0<\lambda<1$ we have

$$\log(1-\lambda)=-\sum_{n=1}^{\infty}\frac{\lambda^n}{n}$$

so that

$$|\log(1-\lambda)|\ge\lambda.$$

Moreover, we have

$$\log(1+\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\lambda^n}{n}\ge\lambda-\frac{\lambda^2}{2}=\lambda\left(1-\frac{\lambda}{2}\right)\ge\frac{\lambda}{2}.$$

When $n=x\pm\langle x\rangle$, we have

$$\left|\log\left(\frac{x}{n}\right)\right|=\left|\log\left(\frac{n\pm\langle x\rangle}{n}\right)\right|=\left|\log\left(1\pm\frac{\langle x\rangle}{n}\right)\right|\ge\frac{1}{2}\frac{\langle x\rangle}{n}\ge\frac{1}{3}\frac{\langle x\rangle}{x}$$

and

$$\frac{1}{\left|\log\left(\frac{x}{n}\right)\right|} = O\left(\frac{x}{\langle x\rangle}\right).$$

For $\frac{2x}{3} < n < \frac{3x}{2}$ and $n \neq x \pm \langle x\rangle$, from $\frac{|x-n|}{n} \leq \frac{1}{2}$ it follows that

$$\begin{aligned}\left|\log\left(\frac{x}{n}\right)\right| &= \left|\log\left(\frac{n+(x-n)}{n}\right)\right| = \left|\log\left(1+\frac{x-n}{n}\right)\right| \\ &\geq \frac{1}{2}\,\frac{|x-n|}{n} \geq \frac{1}{3}\,\frac{\lfloor |x-n| \rfloor}{x}\end{aligned}$$

and

$$\frac{1}{\left|\log\left(\frac{x}{n}\right)\right|} = O\left(\frac{x}{\lfloor |x-n| \rfloor}\right),$$

where $\lfloor |x-n| \rfloor$ is the largest integer not exceeding $|x-n|$. Since two distinct n_1 and n_2 would give two distinct integers $\lfloor |x-n_1| \rfloor$ and $\lfloor |x-n_2| \rfloor$, it follows that

$$\sum_{\substack{\frac{2x}{3}<n<\frac{3x}{2},\\ n\neq x\pm\langle x\rangle}} \frac{1}{\lfloor |x-n| \rfloor} = O\left(\log x\right).$$

Thus

$$\sum_{\substack{\frac{2x}{3}<n<\frac{3x}{2},\\ n\neq x\pm\langle x\rangle}} \frac{1}{\left|\log\left(\frac{x}{n}\right)\right|} = O\left(\sum_{\substack{\frac{2x}{3}<n<\frac{3x}{2},\\ n\neq x\pm\langle x\rangle}} \frac{x}{\lfloor |x-n| \rfloor}\right) = O\left(x\log x\right).$$

Since for $\frac{2x}{3} < n < \frac{3x}{2}$ we have $\Lambda(n) = O\left(\log x\right)$ and $\frac{x}{n} = O\left(1\right)$ and $\left(\frac{x}{n}\right)^c = O\left(1\right)$, it follows that

$$\begin{aligned}&O\left(\sum_{\frac{2x}{3}<n<\frac{3x}{2}} \Lambda(n)\left(\frac{x}{n}\right)^c \frac{1}{T\left|\log\left(\frac{x}{n}\right)\right|}\right) \\ &= O\left(\left(\frac{\log x}{T}\right)\sum_{\frac{2x}{3}<n<\frac{3x}{2}} \frac{1}{\left|\log\left(\frac{x}{n}\right)\right|}\right) \\ &= O\left(\frac{x\left(\log x\right)^2}{T}\right) + O\left(\frac{x\log x}{\langle x\rangle\, T}\right).\end{aligned}$$

Finally we use $\Lambda(x) \leq \log x$ to get

$$O\left(\frac{\Lambda(x)}{T}\right) = O\left(\frac{\log x}{T}\right).$$

Putting everything together, we get

$$\begin{aligned}\int_{c-iT}^{c+iT} \frac{\zeta'(s)}{\zeta(s)} x^s \, \frac{ds}{s} &= \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^s \frac{ds}{s} \\ = \sum_{n<x} \Lambda(n) + O\left(\frac{1}{|c-1|} \frac{x^c}{T}\right) &+ O\left(\frac{x\,(\log x)^2}{T}\right) + \\ &+ O\left(\frac{x \log x}{\langle x \rangle \, T}\right) + O\left(\frac{\log x}{T}\right).\end{aligned}$$

Integrals over the Two Horizontal Line Segments. When we consider the integrals of

$$\frac{-1}{2\pi i} \frac{\zeta'(s)}{\zeta(s)} x^s \frac{ds}{s}$$

over the two horizontal line-segments

$$[-M - iT, \, -M + iT] \quad \text{and} \quad [-M - iT, \, c - iT]\,,$$

we have to worry that the one of the two line segments may contain some zeroes of $\zeta(s)$. To avoid such a situation we are going to replace T by some other appropriate T^* with $|T - T^*| < 1$ by using the following result on the root density of the Riemann zeta function on the critical strip.

Root Density for Riemann Zeta Function on the Critical Strip. For any given $C > 0$ the number of roots ρ of $\zeta(s)$ with $0 \leq \operatorname{Re} \rho \leq 1$ and $|\operatorname{Im} \rho - T| \leq C$ is no more than $O(\log T)$.

From this statement on root density it follows that there exists some T^* such that

$$\begin{cases} |T - T^*| < 1, \\ \\ |\operatorname{Im} \rho - T^*| > \frac{c'}{\log T} \end{cases}$$

for some positive constant c' independent of T. For our use of the contour integration we assume always that such a replacement of T by T^* has been made so that we always have

$$|\operatorname{Im} \rho - T^*| > \frac{c'}{\log T}.$$

To handle the contour integrals over the two horizontal line-segments and the left vertical line-segment, we need the following estimate

$$(\flat)\qquad \frac{\zeta'(s)}{\zeta(s)} = O\left(\log|s|\right) \quad \text{for large} \quad |s|,$$

which we are going to derive from

(i) the expansion

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right) - \gamma$$

with

$$\gamma = \lim_{N\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \log N\right),$$

(ii) the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s),$$

where

$$\chi(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2} - \frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}s\right)},$$

and

(iii) the relation between the sine function and the Gamma functoin

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

$$\zeta(s) = \chi(s)\zeta(1-s),$$

where

$$\chi(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2} - \frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}s\right)}.$$

$$\frac{\chi'(s)}{\chi(s)} = \log\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1}{2} - \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}$$

$$(\natural)\qquad \frac{\zeta'(s)}{\zeta(s)} = \log\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1}{2} - \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)} - \frac{\zeta'(1-s)}{\zeta(1-s)}.$$

From

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

it follows that

$$\frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(1-s)}{\Gamma(1-s)} = \pi \frac{\cos \pi s}{\sin \pi s} = \pi \frac{e^{i\pi\sigma - \pi t} + e^{i\pi\sigma + \pi t}}{e^{i\pi\sigma - \pi t} - e^{i\pi\sigma + \pi t}} = \pi \frac{1 + e^{2i\pi\sigma + \pi t}}{1 - e^{2i\pi\sigma + \pi t}}$$

and

$$\left| \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\Gamma'(1-s)}{\Gamma(1-s)} \right| \leq \pi \frac{e^{\pi|t|} + 1}{e^{\pi|t|} - 1} \quad \text{for } t \neq 0.$$

For $\sigma > 0$ very large, to estimate

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right) - \gamma$$

we can use the comparison with

$$\begin{aligned} \int_{x=1}^{\infty} \left(\frac{1}{x} - \frac{1}{x+|z|} \right) dx &= \log x - \log\left(x + |z|\right) \Big|_{x=1}^{\infty} \\ = \log \frac{x}{x+|z|} \Big|_{x=1}^{\infty} &= \log \frac{1}{1+|z|} = O\left(\log |z|\right). \end{aligned}$$

From ($\natural$) we get

$$\frac{\zeta'(s)}{\zeta(s)} = O\left(\log |s|\right) \quad \text{for large} \ \ |s|.$$

Estimates of Three Integrals. We first consider the integral

$$\int_{M-iT}^{M+iT} \frac{\zeta'(s)}{\zeta(s)} x^s \, \frac{ds}{s},$$

whose limit vanishes as $M \to -\infty$ because of the contribution from the factor x^s and because of the estimate ($\flat$).

The estimate of

$$\int_{M-iT}^{c-iT} \frac{\zeta'(s)}{\zeta(s)} x^s \, \frac{ds}{s} \quad \text{as} \ \ M \to -\infty$$

is completely analogous to that of

$$\int_{M+iT}^{c+iT} \frac{\zeta'(s)}{\zeta(s)} x^s \, \frac{ds}{s} \quad \text{as} \quad M \to -\infty$$

and so we will do the estimate only for the latter integral. From ($\flat$) we get its domination by

$$\int_{\lambda=-\infty}^{c} \log\left(|\lambda| + T\right) \frac{x^\lambda d\lambda}{|\lambda| + T} = O\left(\frac{x^c \, \log T}{T}\right).$$

Computation of Residues. We now compute the residues. The contributions to the residues of

$$-\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s}$$

come from

(i) the pole of $\zeta(s)$ at $s = 1$, which gives rise to x,

(ii) the denominator s at $s = 0$, which gives rise to $-\frac{\zeta'(0)}{\zeta(0)}$,

(iii) the trivial zeroes of $\zeta(s)$ at $-2\,\mathbb{N}$ (where $\mathbb{N}$ is the set of all positive integers), which gives rise to $\sum_{j \in \mathbb{N}} \frac{x^{-2j}}{2j}$,

(iv) the zeroes ρ of $\zeta(s)$ with $|\mathrm{Im}\,\rho| < T$, which gives rise to $-\sum_{\substack{\rho \in Z, \\ |\mathrm{Im}\,\rho| < T}} \frac{x^\rho}{\rho}$.

The sum of all residues is

$$x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{j \in \mathbb{N}} \frac{x^{-2j}}{2j} - \sum_{\substack{\rho \in Z, \\ |\mathrm{Im}\,\rho| < T}} \frac{x^\rho}{\rho}.$$

We simplify

$$\sum_{j \in \mathbb{N}} \frac{x^{-2j}}{2j}$$

to

$$-\frac{1}{2} \log\left(1 - \frac{1}{x^2}\right).$$

Together with the residues we get

$$\sum_{n<x} \Lambda(n) = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\substack{\rho \in Z, \\ |\mathrm{Im}\,\rho| < T}} \frac{x^\rho}{\rho} - \frac{1}{2} \log\left(1 - \frac{1}{x^2}\right) + O\left(\frac{1}{|c-1|} \frac{x^c}{T}\right)$$

$$+O\left(\frac{x\,(\log x)^2}{T}\right)+O\left(\frac{x\log x}{\langle x\rangle\,T}\right)+O\left(\frac{\log x}{T}\right)+O\left(\frac{x^c\,\log T}{T}\right).$$

To handle the denominate $c-1$ in

$$O\left(\frac{1}{|c-1|}\,\frac{x^c}{T}\right)$$

we choose $c = 1+\frac{1}{\log x}$ so that

$$x^c = x^{1+\frac{1}{\log x}} = e^{\left(1+\frac{1}{\log x}\right)\log x} = e^{\log x+1} = ex$$

and

$$\frac{1}{|c-1|}\,\frac{x^c}{T} = \frac{e\,x\,\log x}{T}.$$

With the choice of $c = 1+\frac{1}{\log x}$, the error term $O\left(\frac{x^c\,\log T}{T}\right)$ can be rewritten as $O\left(\frac{x\,\log T}{T}\right)$. Now our final result is the following.

Explicit Formula.

$$\sum_{n<x}\Lambda(n) = x-\frac{\zeta'(0)}{\zeta(0)}-\sum_{\substack{\rho\in Z,\\ |\mathrm{Im}\,\rho|<T}}\frac{x^\rho}{\rho}+O\left(\frac{x\,(\log x)^2}{T}\right)$$
$$+O\left(\frac{x\log x}{\langle x\rangle\,T}\right)+O\left(\frac{x\,\log T}{T}\right).$$