Proof of Functional Equation of Riemann Zeta Function by Mellin Transform, Poisson Summation, and Gauß Error Function

Mellin Transform. The Mellin transform $M(f,s)$ of f is defined by

$$M(f,s)=\int_{\mathbb{R}} f(x)\,|x|^{s-1}\,dx,$$

which is defined for $\alpha<\operatorname{Re}s<\beta$ if $x\mapsto f(x)|x|^{\sigma-1}$ is integrable for $\alpha<\sigma<\beta$.

The Mellin transform is related to the Laplace transform and the Fourier transform. For example, when the support of $f(x)$ is in $(0,\infty]$, with $y=\log x$ we can write

$$M(f,s)=\int_{\mathbb{R}} f(x)\,|x|^{s-1}\,dx=\int_{y\in\mathbb{R}} f\left(e^y\right)e^{sy}dy.$$

Relation Between Mellin Transform and Riemann Zeta Function. Because of the factor $|x|^{s-1}$ in the integrand in the definition of the Mellin transform, the Mellin transform is very much linked to the Riemann zeta function by rescaling x by a factor of n and summing over $n\in\mathbb{N}$. More precisely,

$$\begin{aligned}2\zeta(s)M(f,s)&=\sum_{n\in\mathbb{Z}-0}\frac{1}{|n|^s}\int_{\mathbb{R}} f(x)\,|x|^{s-1}\,dx\\&=\sum_{n\in\mathbb{Z}-0}\int_{\mathbb{R}} f(nx)\,|x|^{s-1}\,dx\\&=2\sum_{n=1}^{\infty}\int_{\mathbb{R}} f(nx)\,|x|^{s-1}\,dx\\&=4\sum_{n=1}^{\infty}\int_{x=0}^{\infty} f(nx)\,x^{s-1}dx\\&=4\sum_{n=1}^{\infty}\left(\int_{x=0}^{1} f(nx)\,x^{s-1}dx+\int_{x=1}^{\infty} f(nx)\,x^{s-1}dx\right)\\&=4\sum_{n=1}^{\infty}\left(\int_{y=1}^{\infty} f\left(\frac{n}{y}\right)\frac{1}{y^{s+1}}dy+\int_{x=1}^{\infty} f(nx)\,x^{s-1}dx\right),\end{aligned}$$

where for the last equality we have used the change of variable $x=\frac{1}{y}$.

Transformation by Use of Poisson Summation Formula. We now use the Poisson summation formula to transform the first term of the last line in the above string of identities. We use $\hat{f}$ to denote the Fourier transform of f with the definition

$$\hat{f}(x) = \int_{\mathbb{R}} f(y) e^{-2\pi i x y} dy.$$

We temporarily use the notation $f_u(x) = f(ux)$. Then

$$\begin{aligned}\widehat{f_u}(x) &= \int_{\mathbb{R}} f_u(y) e^{-2\pi i x y} dy = \int_{\mathbb{R}} f(uy) e^{-2\pi i x y} dy \\ &= \int_{\mathbb{R}} f(y) e^{-2\pi i x \frac{y}{u}} \frac{dy}{u} = \frac{1}{u} \hat{f}\left(\frac{x}{u}\right).\end{aligned}$$

The Poisson summation formula gives

$$\sum_{n \in \mathbb{Z}} f_y(n) = \sum_{n \in \mathbb{Z}} \widehat{f_y}(n),$$

which means

$$\sum_{n \in \mathbb{Z}} f(yn) = \sum_{n \in \mathbb{Z}} \frac{1}{y} \hat{f}\left(\frac{n}{y}\right).$$

When $\hat{f} = f$ and $f(0)$ and $f(-n) = f(n)$, we have

$$\sum_{n=1}^{\infty} f(yn) = \sum_{n=1}^{\infty} \frac{1}{y} f\left(\frac{n}{y}\right)$$

and we can summarize the result of the transformation in the following statement.

Statement. If $\hat{f} = f$ and $f(0)$ and $f(-n) = f(n)$, then

$$\zeta(s) M(f, s) = 2 \sum_{n=1}^{\infty} \int_{x=1}^{\infty} f(nx) \left(x^{s-1} + \frac{1}{x^s}\right) dx,$$

or, to write it in a more symmetric way,

$$(*) \qquad \zeta(s) M(f, s) = 2 \sum_{n=1}^{\infty} \int_{x=1}^{\infty} f(nx) \left(x^{s-\frac{1}{2}} + x^{\frac{1}{2}-s}\right) x^{-\frac{1}{2}} dx.$$

Example of Even Function Vanishing at the Origin and Equal to Own Fourier Transform. We now apply this Mellin-transform representation of the Riemann zeta function to the special case where the function is a modification of the Gauß error function $e^{-\pi x^2}$. Since $e^{-\pi x^2}$ is equal to its own Fourier transform (because

$$\int_{\mathbb{R}} e^{-\pi y^2} e^{-2\pi ixy} dy = \int_{\mathbb{R}} e^{-\pi(y+ix)^2} e^{-\pi x^2} dy$$

and

$$\int_{\mathbb{R}} e^{-\pi(y+ix)^2} e^{-\pi x^2} dy = \int_{\mathbb{R}} e^{-\pi y^2} dy = 1$$

by contour integration and Cauchy's theorem) and since

$$\left(3\left(\frac{1}{2\pi i}\,\frac{d}{dx}\right)^2 - 2\pi\left(\frac{1}{2\pi i}\,\frac{d}{dx}\right)^4\right) e^{-\pi x^2} = \left(3x^2 - 2\pi x^4\right) e^{-\pi x^2},$$

it follows that the function

$$f(x) := \left(3x^2 - 2\pi x^4\right) e^{-\pi x^2}$$

satisfies the conditions that $f(0) = 0$, $(-x) = f(x)$, and $\hat{f} = f$.

In order to apply to this particular function $f(x)$, the relation which we derived between Mellin transform and the Riemann zeta function, we have to first compute the Mellin transform $M(f, s)$ of this particular function

$$f(x) = \left(3x^2 - 2\pi x^4\right) e^{-\pi x^2}.$$

From the definition of the Mellin transform we have

$$M(f, s) = 3\,M\left(e^{-\pi x^2},\, s+2\right) - 2\pi\,M\left(e^{-\pi x^2},\, s+4\right).$$

The Mellin transform $M\left(e^{-\pi x^2},\, s\right)$ of the function $e^{-\pi x^2}$ can be computed directly from definition as follows.

$$M\left(e^{-\pi x^2},\, s\right) = \int_{\mathbb{R}} e^{-\pi x^2}\, |x|^{s-1}\, dx$$

which, with the change of variables $y = \pi x^2$, becomes

$$\int_{y=0}^{\infty} e^{-y}\, y^{\frac{s}{2}-1}\, \pi^{-\frac{s}{2}}\, dy = \pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right).$$

From $\Gamma(x+1) = x\Gamma(x)$ we conclude that

$$M\left(f,s\right) = 3\pi^{-\frac{s+2}{2}}\,\Gamma\left(\frac{s+2}{2}\right) - 2\pi\,\pi^{-\frac{s+4}{2}}\,\Gamma\left(\frac{s+4}{2}\right)$$

$$= M\left(3\pi^{-1}\frac{s}{2} - 2\pi\,\pi^{-2}\left(\frac{s}{2}+1\right)\pi^{-\frac{s}{2}}\right)\Gamma\left(\frac{s}{2}\right) = s(1-s)\frac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{2\pi}.$$

We now rewrite $(*)$, for our particular choice of $f(x) = \left(3x^2 - 2\pi x^4\right)e^{-\pi x^2}$, as

$$(\dagger) \qquad s(1-s)\frac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{2\pi}\zeta(s) = 2\sum_{n=1}^{\infty}\int_{x=1}^{\infty} f\left(nx\right)\left(x^{s-\frac{1}{2}} + x^{\frac{1}{2}-s}\right)x^{-\frac{1}{2}}dx.$$

Since the right-hand side is unchanged with the transformation $s \to 1-s$, we have the functional equation of the Riemann zeta function which is simply the invariance of the left-hand side under the transformation $s \to 1-s$ (after the removal of the factor 2π from the denominator).

$$s(1-s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = s(1-s)\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Estimation of Functional Value on Vertical Line From That on Intersection Point with Real Axis. One important property of the function $f(x) = \left(3x^2 - 2\pi x^4\right)e^{-\pi x^2}$ is that $f(x) < 0$ for $x \geq 1$, because $3 < 2\pi$. Let $\sigma = \operatorname{Re} s$. Then

$$\left|2\sum_{n=1}^{\infty}\int_{x=1}^{\infty} f\left(nx\right)\left(x^{s-\frac{1}{2}} + x^{\frac{1}{2}-s}\right)x^{-\frac{1}{2}}dx\right|$$

$$\leq -2\sum_{n=1}^{\infty}\int_{x=1}^{\infty} f\left(nx\right)\left(x^{\sigma-\frac{1}{2}} + x^{\frac{1}{2}-\sigma}\right)x^{-\frac{1}{2}}dx.$$

Thus

$$(\sharp) \qquad \left|s(1-s)\,\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s)\right| \leq -\,\sigma(1-\sigma)\,\pi^{-\frac{\sigma}{2}}\,\Gamma\left(\frac{\sigma}{2}\right)\,\zeta(\sigma).$$

This is important for growth estimates of the Riemann zeta function on vertical lines which intersect the real axis at points of $(-\infty, 1]$, because of the symmetry of

$$s(1-s)\,\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s)$$

from the functional equation. It reduces the problem of such estimates to the growth behavior of the Gamma function which can be computed from the Stirling formula.

Definition of Xi Function. We now define the xi function by

$$\xi(s) = -\frac{s(1-s)}{2}\frac{\Gamma\left(\frac{s}{2}\right)\zeta(s)}{\pi^{\frac{s}{2}}}.$$

The functional equation of the Riemann zeta function is simply

$$\xi(1-s) = \xi(s).$$