Gamma Function

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Definition of Gamma Function. The Gamma function in a real variable is defined by

$$\Gamma\left(x\right)=\int_0^\infty t^{x-1}e^{-t}dt$$

for $x>0$ to make sure that the integral converges at $t=0$. When $x>1$, by integration by parts we get

$$\Gamma\left(x\right)=\left[-t^{x-1}e^{-t}\right]_{t=0}^{t=\infty}+\left(x-1\right)\int_0^\infty t^{x-2}e^{-t}dt=\left(x-1\right)\Gamma\left(x-1\right).$$

From $\Gamma(1)=\int_0^\infty e^{-t}=1$ it follows that

$$\Gamma\left(n\right)=\left(n-1\right)!\ .$$

So the Gamma function is the generalization of the factorial function from integer values to real values. The defining formula

$$\Gamma\left(z\right)=\int_0^\infty t^{z-1}e^{-t}dt$$

actually defined $\Gamma(z)$ for $z\in\mathbb{C}$ with $\operatorname{Re}z>0$.

Beta Function. A similar analog of the generalization of the binomial coefficient

$$\binom{m+n}{m}=\frac{(m+n)!}{m!\,n!}$$

is the Beta function defined by

$$B\left(x,y\right)=\frac{\Gamma\left(x+y\right)}{\Gamma(x)\,\Gamma(y)}.$$

We are going to derive the formula for the Beta function as a definite integral whose integrand depends on the variables x and y. This is done by reversing the order of integration of a double integral. For $x>0$ and $y>0$ we have

$$\Gamma(x)\Gamma(y)=\left(\int_0^\infty t^{x-1}e^{-t}dt\right)\left(\int_0^\infty u^{y-1}e^{-u}du\right).$$

Using the transformation $u=tv$ and then the transformation $w=t\left(1+v\right)$, we obtain

$$\begin{aligned}\Gamma(x)\Gamma(y)&=\left(\int_0^\infty t^{x-1}e^{-t}dt\right)\left(\int_0^\infty t^{y}v^{y-1}e^{-tv}dv\right)\\&=\left(\int_0^\infty v^{y-1}dv\right)\left(\int_0^\infty t^{x+y-1}e^{-t(1+v)}dv\right)\\&=\left(\int_0^\infty v^{y-1}dv\right)\left(\int_0^\infty \frac{w^{x+y-1}e^{-w}dw}{(1+v)^{x+y}}\right)\\&=\Gamma\left(x+y\right)\int_0^\infty \frac{v^{y-1}dv}{(1+v)^{x+y}},\end{aligned}$$

from which it follows that

$$B\left(x,y\right)=\int_0^\infty \frac{v^{y-1}dv}{(1+v)^{x+y}}.$$

Finally we use the transformation $v=\frac{\lambda}{1-\lambda}$ to get the alternative formulation

$$B\left(x,y\right)=\int_0^1 \lambda^{x-1}\left(1-\lambda\right)^{y-1}d\lambda,$$

which is symmetric in x and y.

Relation Between Gamma Function and Sine Function. A very useful case for the Beta function is when $x+y=1$ in the above formula, in which case

$$\Gamma(x)\Gamma(1-x)=\int_0^\infty \frac{v^{x-1}dv}{1+v},$$

which by residue calculus applied to the function

$$\frac{z^{x-1}dz}{1+z}$$

integrated over the contour integral of the boundary of the domain

$$\{\, r<|z|<R \,\} - \{\, \operatorname{Re} z \geq 0,\ -r \leq \operatorname{Im} z \leq r \,\},$$

yields

$$\frac{\pi}{\sin \pi x}.$$

Thus we have the following important formula relating the gamma function to the sine function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

for $0 < x < 1$.

Duplication Formula for Gamma Function. Another consequence of the Beta function is the *duplication formula.* By using $\lambda = \sin^2\theta$, we get

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \int_{\lambda=0}^{1} \frac{d\lambda}{\sqrt{\lambda(1-\lambda)}}$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \frac{d\left(\sin^2\theta\right)}{\sqrt{\sin^2\theta\,\cos^2\theta}} = 2\int_{\theta=0}^{\frac{\pi}{2}} d\theta = \pi.$$

Thus $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Using the substitution $\lambda = \frac{1}{2} - \frac{1}{2}\sqrt{\mu}$ so that $\lambda(1-\lambda) = \frac{1}{4} - \frac{1}{4}\mu$ and $d\lambda = -\frac{d\mu}{4\sqrt{\mu}}$, we get

$$B\left(x,x\right) = \int_{\lambda=0}^{1} \lambda^{x-1}(1-\lambda)^{y-1}d\lambda$$

$$= 2\int_{\lambda=0}^{\frac{1}{2}} \lambda^{x-1}(1-\lambda)^{y-1}d\lambda = \frac{1}{2}\int_0^1 \left(\frac{1}{4}-\frac{1}{4}\mu\right)^{x-1} \mu^{-\frac{1}{2}}$$

$$= 2^{1-2x}\int_0^1 \left(1-\mu\right)^{x-1} \mu^{-\frac{1}{2}}d\mu = 2^{1-2x}\, B\left(x,\frac{1}{2}\right).$$

Thus we have the following *duplication formula.*

$$\Gamma\left(2x\right)\Gamma\left(\frac{1}{2}\right) = 2^{2x-1}\,\Gamma\left(x\right)\Gamma\left(x+\frac{1}{2}\right).$$

Meromorphic Extension of Gamma Function by Representation by Contour Integral. We now look at the meromorphic extension of $\Gamma(z)$ by representing the gamma function as a contour integral. Let C be a contour which starts from positive infinity and goes toward the origin above the real axis and circles the origin once in the positive direction and then goes to positive infinity below the real axis. Then for $\operatorname{Re} z > 0$

$$\Gamma(z) = \int_C e^{-w}(-w)^{z-1}dw.$$

The precise evaluation of the integral is as follows.

$$(-w)^{z-1} = e^{(z-1)\log(-w)}$$

with

$$\log(-w) = \log\rho + i(\phi - \pi)$$

when $w = \rho e^{i\phi}$ and $0 < \phi < 2\pi$. So

$$\begin{aligned}\int_C e^{-w}(-w)^{z-1}dw &= \int_{\rho=\infty}^{0} e^{-\rho}e^{(z-1)(\log\rho - i\pi)}d\rho + \int_{\rho=0}^{\infty} e^{-\rho}e^{(z-1)(\log\rho + i\pi)}d\rho \\ &= \left(e^{(z-1)\pi i} - e^{-(z-1)\pi i}\right)\int_{\rho=0}^{\infty} e^{-\rho}e^{(z-1)\log\rho}d\rho \\ &= 2i\sin\left((z-1)\pi\right)\int_{\rho=0}^{\infty} e^{-\rho}\rho^{z-1}d\rho \\ &= -2i\sin\pi z\Gamma(z).\end{aligned}$$

Note that the condition $\operatorname{Re} z > 0$ is used to make sure that the integral around a small circle centered at the origin goes to zero as the radius of the circle goes to zero. When the condition $\operatorname{Re} z > 0$ is not assumed we can define the Gamma function by

$$\Gamma(z) = \frac{i}{2\sin\pi z}\int_C e^{-w}(-w)^{z-1}dw.$$

The only possible poles of $\Gamma(z)$ are the zeroes of $\sin\pi z$, namely $z \in \mathbb{Z}$. We already know that $\Gamma(z)$ is regular at points of $\operatorname{Re} z > 0$. Thus the only possible poles of $\Gamma(z)$ are at the non-positive integers. The residues at a non-positive integer $z = -n$ is computed as follows. Since the principal part of

$$e^{-w}(-w)^{n-1} = (-w)^{-n-1}\sum_{k=0}^{\infty}\frac{(-1)^k w^k}{k!}$$

at $w = 0$ is $-\frac{1}{n!\,w}$, it follows that

$$\int_C e^{-w}(-w)^{-n-1}dw = -\frac{2\pi i}{n!}$$

and the residue of $\Gamma(z)$ at $z = -n$ is equal to

$$\lim_{z \to -n} -\frac{2\pi i}{n!}\,\frac{i\,(z+n)}{2\sin \pi z} = \frac{(-1)^n}{n!}.$$

After we have the meromorphic extension of $\Gamma(z)$, the recurrent formula

$$\Gamma\left(z\right) = \left(z-1\right)\Gamma\left(z-1\right),$$

the relation between the gamma function and the sine function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

and the duplication formula

$$\Gamma\left(2z\right)\Gamma\left(\frac{1}{2}\right) = 2^{2z-1}\,\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right).$$

all hold for all $z \in \mathbb{C}$. Of course, conversely any one of these three formulas can be used to define the meromorphic extension of $\Gamma(z)$ instead of using the above representation of the gamma function as a contour integral.

The pole-set and the residues of $\Gamma(z)$, which was computed earlier from the representation of $\Gamma(z)$ by a contour integral, can also be obtained by using the relation between the gamma function and the sine function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Infinite Product Expansion of Gamma Function. We now consider the infinite product expansion of $\Gamma(z)$. Using the Beta function we have

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \int_0^1 (1-t)^{z-h-1}t^{h-1}dt.$$

For $\operatorname{Re} z > h > 0$ with z fixed and h variable, we have the principal part $\frac{1}{h}$ of

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}.$$

We would like to take it out so that what is left will be holomorphic for h in a small open neighborhood of 0 in $\mathbb{C}$. For that purpose, we use

$$\frac{1}{h}=\int_0^1 t^{h-1}dt$$

to get

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}=\int_0^1 (1-t)^{z-h-1}t^{h-1}dt=\frac{1}{h}+\int_0^1 \left((1-t)^{z-h-1}-1\right)t^{h-1}dt.$$

The integral on the right-hand side now is holomorphic for h in a small open neighborhood of 0 in $\mathbb{C}$ and we can write it as its value at $h=0$ plus a term of the order $o(h)$ as $h\to 0$. Thus

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}=\frac{1}{h}+\int_0^1 \left((1-t)^{z-1}-1\right)t^{-1}dt+o(h).$$

This is the Laurent expansion of the Beta function $B\left(z-h,z\right)$ in the variable h at $h=0$.

We compare this to the Laurent series expansion of

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}$$

in h and get

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)}=\frac{1}{\Gamma(z)}\left(\Gamma(z)-h\,\Gamma'(z)+\cdots\right)\left(\frac{1}{h}+A+\cdots\right),$$

where A is a constant. Equating the constant terms of

$$\frac{1}{\Gamma(z)}\left(\Gamma(z)-h\,\Gamma'(z)+\cdots\right)\left(\frac{1}{h}+A+\cdots\right)$$
$$=\frac{1}{h}+\int_0^1 \left((1-t)^{z-1}-1\right)t^{-1}dt+o(h),$$

we get

$$\frac{\Gamma'(z)}{\Gamma(z)}=\int_0^1 \left(1-(1-t)^{z-1}\right)t^{-1}dt-A$$

for $\operatorname{Re} z > 0$. Using

$$\frac{1}{t} = \frac{1}{1-(1-t)} = \sum_{n=0}^{\infty} (1-t)^n \,,$$

we get

$$\begin{aligned}\frac{\Gamma'(z)}{\Gamma(z)} &= -A + \int_0^1 \left(1-(1-t)^{z-1}\right)\left(\sum_{n=0}^{\infty}(1-t)^n\right) dt \\ &= -A + \int_0^1 \left(\sum_{n=0}^{\infty}\left((1-t)^n - (1-t)^{n+z-1}\right)\right) dt \\ &= -A + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right).\end{aligned}$$

We can rewrite it as

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right) - C$$

for some constant C. To determine the constant C, we integrate and take exponents of both sides and get

$$\frac{1}{\Gamma(z)} = e^{Cz} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}.$$

Setting $z = 1$, we get

$$1 = e^{C} \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right) e^{-\frac{1}{n}}.$$

Hence

$$C = -\log \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right) e^{-\frac{1}{n}} = \lim_{N\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N} - \log N\right)$$

which is equal to the Euler constant γ. We have finally the following infinite product decomposition for $\Gamma(z)$.

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}.$$

Stirling's Formula. Stirling's formula gives the asymptotic behavior of $\Gamma(z)$ for large z. Its formulation is

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log (2\pi) + O\left(\frac{1}{|z|}\right)$$

for z in the the set

$$-\pi + \delta \leq \arg z \leq \pi - \delta$$

with $\delta > 0$.

To derive Stirling's formula we introduce an auxiliary function

$$\psi(v) = \lfloor v \rfloor - v + \frac{1}{2}$$

and its indefinite integral. Some integral constructed from $\psi(v)$ will serve as a bridge between the growth behavior of $\Gamma(z)$ for large z and the growth behavior of $n!$ for large n. Then Striling's formula will be derived from the the growth behavior of $n!$ for large n, which in turn will be derived by comparing the logarithm of $n!$ with the integral of the logarithmic function over the $\left[\frac{1}{2}, n + \frac{1}{2}\right]$.

The function $\psi(v)$ has period 1 and on the interval $[0, 1)$ it is the same as the function $y = -x + \frac{1}{2}$ whose graph is the straight line joining the point $\left(0, \frac{1}{2}\right)$ to the point $\left(1, -\frac{1}{2}\right)$. Moreover, the graph of $\psi(v)$ is invariant under translation by an integer in its variable. Let

$$\phi(u) = \int_0^u \psi(v) dv$$

which on the interval $[0, 1]$ is the same as the function $y = \frac{1}{2}\left(x - x^2\right)$ and whose graph is the inverted parabola which starts at the origin and increases to its maximum of $\frac{1}{8}$ at the point $x = \frac{1}{2}$ and then decreases to 0 at $x = 1$. Moreover, the graph of $\phi(u)$ is invariant under translation by an integer in its variables.

The two functions $\phi(u)$ and $\psi(v)$ are linked by $\phi' = \psi$. One key estimate in the proof of Stirling's formula which we present here is an integration by parts applied to $\frac{\psi(u)}{u+z}$ over $0 \leq u < \infty$ for z in the the set

$$-\pi + \delta \leq \arg z \leq \pi - \delta$$

which is the closed sector excluding points whose radius vector makes an angle less than δ with the positive axis for some fixed positive number δ. Let $r = |z|$. Then

$$\int_{u=0}^{\infty} \frac{\psi(u)du}{u+z} = \int_{u=0}^{\infty} \frac{\phi(u)du}{(u+z)^2}$$
$$= O\left(\int_{u=0}^{\infty} \frac{du}{|u+z|^2}\right) = O\left(\int_{u=0}^{\infty} \frac{du}{u^2 + r^2 - 2ur\cos\delta}\right)$$
$$= O\left(\int_{u=0}^{\infty} \frac{du}{(u - r\cos\delta)^2 + \left(\sqrt{1-\cos^2\delta}\, r\right)^2}\right)]$$
$$= \left[\frac{1}{\sqrt{1-\cos^2\delta}\, r} \arctan\left(\frac{u - r\cos\delta}{\sqrt{1-\cos^2\delta}\, r}\right)\right]_{u=0}^{u=\infty}$$
$$= \frac{1}{\sqrt{1-\cos^2\delta}\, r} \arctan\left(\frac{r\cos\delta}{\sqrt{1-\cos^2\delta}\, r}\right) = O\left(\frac{1}{r}\right).$$

We now evaluate the integral

$$\int_{u=0}^{\infty} \frac{\psi(u)du}{u+z} = \lim_{N\to\infty} \int_{u=0}^{N} \frac{\psi(u)du}{u+z}$$

directly to link it to the Gamma function $\Gamma(z)$.

$$\int_{u=0}^{N} \frac{\psi(u)du}{u+z} = \int_{u=0}^{N} \frac{\lfloor v \rfloor - v + \frac{1}{2}}{u+z}\, du$$
$$= \sum_{n=0}^{N-1} \int_{u=n}^{n+1} \frac{\lfloor v \rfloor - v + \frac{1}{2}}{u+z}\, du = \sum_{n=0}^{N-1} \int_{u=n}^{n+1} \left(\frac{n + \frac{1}{2} + z}{u+z} - 1\right) du$$
$$= -N + \sum_{n=0}^{N-1} \int_{u=n}^{n+1} \frac{n + \frac{1}{2} + z}{u+z}\, du = -N + \sum_{n=0}^{N-1} \left(n + \frac{1}{2} + z\right) \left[\log(u+z)\right]_{u=n}^{n+1}$$
$$= -N + \sum_{n=0}^{N-1} \left(\left(n + \frac{1}{2} + z\right) \log(n+1+z) - \left(n + \frac{1}{2} + z\right) \log(n+z)\right)$$
$$= -N + \left(N - \frac{1}{2} + z\right) \log(N+z) - \left(\frac{1}{2} + z\right) \log z$$
$$+ \sum_{n=1}^{N-1} \left(-\left(n + \frac{1}{2} + z\right) \log(n+z) + \left(n - 1 + \frac{1}{2} + z\right) \log(n+z)\right)$$

$$= -N + \left(N - \frac{1}{2} + z\right) \log\left(N + z\right) - \left(\frac{1}{2} + z\right) \log z - \sum_{n=1}^{N-1} \log(n + z)$$

$$= -N + \left(N - \frac{1}{2} + z\right) \log\left(N + z\right) - \left(\frac{1}{2} + z\right) \log z$$

$$+ \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - z \sum_{n=1}^{N-1} \frac{1}{n} - \log\left((N-1)!\right)$$

$$= -N + \left(N - \frac{1}{2} + z\right) \log\left(N + z\right) - \left(\frac{1}{2} + z\right) \log z$$

$$+ \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right) - \gamma z - z \log N - \log\left((N-1)!\right) + o(1),$$

where the term $o(1)$ means having limit 0 as $N \to \infty$ for fixed z. We are going to use

$$(\dagger) \qquad \log\left((N-1)!\right) = -N + \left(N - \frac{1}{2}\right) \log N + C + o(1)$$

(where C is a constant) without proof and then verify it later. First we use the duplication formula and $\log\left((N-1)!\right) = \log \Gamma\left(N\right)$ to determine the constant C. For fixed z we have

$$-N + \left(N - \frac{1}{2} + z\right) \log\left(N + z\right) - z \log N - \log\left((N-1)!\right) + C = z + o(1)$$

as $N \to \infty$, because

$$\log\left(N + z\right) = \log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right).$$

It follows from the logarithm

$$\log \Gamma(z) = -\gamma\, z - \log z + \lim_{N\to\infty} \sum_{n=1}^{N-1} \left(\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right)$$

of the product formula of $\Gamma(z)$ that

$$\Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + C + O\left(\frac{1}{|z|}\right)$$

for z in the the set

$$-\pi + \delta \leq \arg z \leq \pi - \delta$$

with $\delta > 0$. We now use the duplication formula

$$\Gamma\left(2z\right)\Gamma\left(\frac{1}{2}\right) = 2^{2z-1}\,\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right).$$

of $\Gamma(z)$ to determine the constant C. From

$$\begin{aligned}&\left(2z-\frac{1}{2}\right)\log\left(2z\right) - 2z + C + O\left(\frac{1}{|z|}\right) + \log\sqrt{\pi}\\ &= (2z-1)\log 2 + \left(z-\frac{1}{2}\right)\log z - z + C + z\log z - z - \frac{1}{2} + C\end{aligned}$$

we conclude that $C = \frac{1}{2}\log\left(2\pi\right)$. This concludes the proof of Stirling's formula that

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log\left(2\pi\right) + O\left(\frac{1}{|z|}\right)$$

for z in the the set

$$-\pi + \delta \leq \arg z \leq \pi - \delta$$

with $\delta > 0$.

We now return to the proof of the identity (†).

$$\begin{aligned}&\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \log t\,dt = \int_0^{\frac{1}{2}} \left(\log(\nu+t) + \log(\nu-t)\right)dt\\ &= \int_0^{\frac{1}{2}} \left(\log\left(\nu^2\right) + \log\left(1-\frac{t^2}{\nu^2}\right)\right)dt = \log\nu + C_\nu,\end{aligned}$$

where $|C_\nu| \leq \frac{A}{\nu^2}$ for some constant A independent of ν. Then

$$\log\left((N-1)!\right) = \int_{\frac{1}{2}}^{N-\frac{1}{2}} \log t\,dt - \sum_{\nu=1}^{N-1} C_\nu = \left(N-\frac{1}{2}\right)\log N - N + C + o(1)$$

for some constant C as $N \to \infty$.