Growth Rate of Riemann Zeta Function on Vertical Lines

Define

$$\mu(\sigma)=\inf\left\{a\in\mathbf{R}\;\middle|\;\zeta(\sigma+it)=O\left(|t|^a\right)\right\}.$$

We are going to give the growth rate of the Riemann zeta function on vertical lines. It will done with the use of the functional equation, Stirling's formula, and the theorem of Phragmén-Lindelöf on the interpolation of growth rates on vertical lines. We will first use the functional equation and Stirling's formula to treat the case where the abscissa of the vertical line is outside the interval $[0,1]$. Then we will prove the theorem of Phragmén-Lindelöf on the interpolation of growth rates on vertical lines and use it to treat the case where the abscissa of the vertical line is in the interval $[0,1]$.

Proposition (Growth Rate of Zeta Function on Vertical Lines with Abscissa outside [0, 1]*).*

$$\begin{cases}\mu(\sigma)=0 \text{ if } \sigma>1,\\[2ex] \mu(\sigma)=\frac{1}{2}-\sigma \text{ if } \sigma<0.\end{cases}$$

Proof. The case of $\sigma>1$ is clear, because $|\zeta(s)|=O(1)$ for $\sigma=\operatorname{Re}s>1$ from the definition of ζ as an infinite series. We are to verify the case $\sigma<0$ from the functional equation of the Riemann zeta function. The functional equation of the Riemann zeta function is $\xi(1-s)=\xi(s)$, where

$$\xi(s)=\frac{1}{2}s\,(s-1)\,\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s).$$

We can rewrite it as

$$\zeta(s)=\chi(s)\zeta(1-s),$$

where

$$\chi(s)=\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}s\right)}.$$

There are the following two other useful forms of $\chi(s)$ which are obtained by using the duplication formula for the Gamma function and the relation between the Gamma function and the sine function.

$$\chi(s)=2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s),$$

$$\chi(s) = 2^{s-1}\pi^s \frac{\sec\frac{\pi s}{2}}{\Gamma(s)}.$$

We are going to use Stirling's formula. Since

$$\log\Gamma(\sigma+it) = \left(\sigma+it-\frac{1}{2}\right)\log(\sigma+it) - (\sigma+it) + \frac{1}{2}\log 2\pi + O\left(\frac{1}{t}\right)$$

$$= \left(\sigma+it-\frac{1}{2}\right)\log(it) - it + \left(\left(\sigma+it-\frac{1}{2}\right)\log\left(\frac{\sigma+it}{it}\right) - \sigma\right) + \frac{1}{2}\log 2\pi + O\left(\frac{1}{t}\right)$$

and

$$\left(\sigma+it-\frac{1}{2}\right)\log\left(\frac{\sigma+it}{it}\right) - \sigma = \left(\sigma+it-\frac{1}{2}\right)\log\left(1-i\frac{\sigma}{t}\right) - \sigma$$

$$= \left(\sigma+it-\frac{1}{2}\right)\left(-\sum_{\ell=1}^{\infty}\left(i\frac{\sigma}{t}\right)^{\ell}\right) - \sigma$$

$$= \left(\sigma-\frac{1}{2}\right)\left(-\sum_{\ell=1}^{\infty}\left(i\frac{\sigma}{t}\right)^{\ell}\right) + (it)\left(-\sum_{\ell=2}^{\infty}\left(i\frac{\sigma}{t}\right)^{\ell}\right) = O\left(\frac{1}{t}\right),$$

it follows that

$$\log\Gamma(\sigma+it) = \left(\sigma+it-\frac{1}{2}\right)\log(it) - it + \frac{1}{2}\log 2\pi + O\left(\frac{1}{t}\right).$$

Because

$$\log(it) = \log t + i\frac{\pi}{2},$$

and

$$\exp\left(\left(\sigma+it-\frac{1}{2}\right)\log(\sigma+it)\right) = t^{\sigma+it-\frac{1}{2}}\, e^{i\frac{\pi}{2}\left(\sigma+it-\frac{1}{2}\right)}$$

$$= t^{\sigma+it-\frac{1}{2}}\, e^{i\frac{\pi}{2}\left(\sigma-\frac{1}{2}\right)} e^{-\frac{\pi}{2}t},$$

we conclude that

$$\Gamma(\sigma+it) = t^{\sigma+it-\frac{1}{2}} e^{-\frac{1}{2}\pi t - it + \frac{1}{2}i\pi\left(\sigma-\frac{1}{2}\right)}\sqrt{2\pi}\left(1+O\left(\frac{1}{t}\right)\right).$$

It follows from

$$\sec\frac{\pi s}{2} = \frac{2}{e^{i\frac{\pi(\sigma+it)}{2}} + e^{-i\frac{\pi(\sigma+it)}{2}}}$$

$$= \frac{2}{e^{i\frac{\pi\sigma}{2}-\frac{\pi t}{2}} + e^{-i\frac{\pi\sigma}{2}+\frac{\pi t}{2}}}$$
$$= 2e^{\frac{i\pi\sigma}{2}-\frac{\pi t}{2}}\left(1 + O\left(\frac{1}{t}\right)\right)$$

and

$$\chi(s) = 2^{s-1}\pi^s \frac{\sec\frac{\pi s}{2}}{\Gamma(s)}$$

that

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-\frac{1}{2}} e^{i\left(t+\frac{\pi}{4}\right)}\left(1 + O\left(\frac{1}{t}\right)\right)$$

and

$$|\chi(s)| = \left(\frac{2\pi}{t}\right)^{\sigma-\frac{1}{2}}\left(1 + O\left(\frac{1}{t}\right)\right).$$

Since $|\zeta(1-s)| = O(1)$ for fixed $\sigma < 0$, it follows from

$$\zeta(s) = \chi(s)\zeta(1-s)$$

that

$$|\zeta(\sigma + it)| = O\left(|t|^{\frac{1}{2}-\sigma+\epsilon}\right)$$

for $\sigma < 0$ and ϵ. Q.E.D.

We are going to use the following theorem of Phragmén-Lindelöf to get the growth rate of the Riemann zeta function on vertical lines with abscissa in $[0, 1]$.

Theorem (Phragmén-Lindelöf) on Interpolation of Growth Rates on Vertical Lines. Let $s = \sigma + it$ and $\phi(s)$ be holomorphic and of the order $O\left(e^{\epsilon|t|}\right)$ for any $\epsilon > 0$ and $\sigma_1 \leq \sigma \leq \sigma_2$. Assume that $\phi(\sigma_1 + it) = O\left(|t|^{k_1}\right)$ and $\phi(\sigma_2 + it) = O\left(|t|^{k_2}\right)$. Let $k(\sigma)$ be a linear function of σ such that $k(\sigma_1) = k_1$ and $k(\sigma_2) = k_2$. Then $\phi(\sigma + it) = O\left(|t|^{k(\sigma)}\right)$.

Proof. Assume first that $k_1 = k_2 = 0$ and that $\phi(s)$ is bounded on $\{\sigma = \sigma_1\}$ and $\{\sigma = \sigma_2\}$ by, say, M. Let $g(s) = e^{\epsilon s i}\phi(s)$. Then

$$|\phi(s)| = e^{-\epsilon t}|\phi(s)| \leq |\phi(s)| \leq M$$

for $t \geq 0$ and either $\sigma = \sigma_1$ or $\sigma = \sigma_1$. Moreover, $|g(s)| \to 0$ as $t \to \infty$. Hence $|g(s)| \leq M$ on the rectangle with sides (σ_1, σ_2) and $(0, T)$ on the two axes for T sufficiently large. Hence $|\phi(s)| \leq e^{\epsilon t} M$. Let $\epsilon \to 0$ and we get $|\phi(s)| \leq M$ for $t \geq 0$. Likewise, we get the same conclusion for $t \leq 0$. For the general case, we let

$$\psi(s) = (-is)^{k(s)} = e^{k(s) \log(-is)},$$

where the branch of $\log(-is)$ is taken with $-\pi < \arg s < \pi$ and for our purpose of studying growth rates we can confine ourselves to the value of the variable s with $|\operatorname{Im} s| \geq 1$. Then $|\psi(s)| = t^{k(\sigma)+O(1)}$ and we use $\Phi(s) = \frac{\phi(s)}{\psi(s)}$ instead of $\phi(s)$. Q.E.D.

Theorem (Growth Rate of Zeta Function on Vertical Lines).

$$\begin{cases} \mu(\sigma) = 0 \text{ if } \sigma > 1, \\ \\ \mu(\sigma) = \frac{1}{2} - \sigma \text{ if } \sigma < 0, \\ \\ \mu(\sigma) = \frac{1}{2}(1 - \sigma) \text{ if } 0 \leq \sigma \leq 1. \end{cases}$$

Proof. Only the case $0 \leq \sigma \leq 1$ needs verification. From

$$\left(1 - \frac{1}{2^{s-1}}\right) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

and $\zeta(s) = \chi(s)\zeta(1-s)$ and the estimate

$$|\chi(s)| = \left(\frac{2\pi}{t}\right)^{\sigma - \frac{1}{2}} \left(1 + O\left(\frac{1}{t}\right)\right)$$

it follows that $|\zeta(s)| = O\left(e^{\varepsilon|t|}\right)$ for any $\varepsilon > 0$ and $1 \leq \sigma \leq 1$ as $|t| \to \infty$ when s is written as $\sigma + it$. The verification of the case $0 \leq \sigma \leq 1$ now follows from the theorem of Phragmén-Lindelöf on the interpolation of growth rates on vertical lines. Q.E.D.