CONSTRUCTION OF HARMONIC FUNCTIONS
WITH SINGULARITIES ON RIEMANN SURFACES
AS ELECTROSTATIC POTENTIALS

§1.*Motivation from Physics*

We start out with a Riemann surface M which may be compact or non-compact. As a means of

 (i) constructing holomorpihc functions, in the case of M being noncompact simply connected, in order to prove the uniformization theorem, or

 (ii) constructing holomorpihc forms and meromorphic forms with certain prescribed singularities, in the case of M being compact, in order to prove the theorems of Riemann-Roch and Abel,

we use the physical motivation of electrostatic fields from point charges, dipoles, and multiple-poles to construct harmonic functions on M with certain prescribed singularities by minimizing Dirichlet integrals

A harmonic function is a function satisfying the Laplace equation and the Laplace equation is the Euler-Lagrange equation for the variational problem of minimizing the global square norm of the gradient. The global square norm of the gradient is known as the *Dirichlet integral.* We will give a precise definition later. If we just want to minimize the Dirichlet integral in the case of a compact Riemann surface, we clearly would get a constant function as the answer, which is completely useless. One way to avoid getting a constant function is to prescribe boundary conditions.

On a compact Riemann surface or an abstract noncompact Riemann surface we have no explicit boundary. So we have to try something else. We try to minimize the Dirichlet integral with the condition that the solution has the singularity of the real part of a meromorphic function with a simple pole at a chosen point of the Riemann surface. In a way this is a boundary condition which is a growth condition at an artifically created singleton boundary.

Originally this method was used to construct a meromorphic function on a simply connected noncompact Riemann surface which maps the Riemann surface biholomorphically onto the Riemann sphere minus a point or a curve segment. We expect this harmonic function with a simple pole singularity which minimizes the Dirichlet integral to give us such a meromorphic function. The motivation for such a strong conclusion comes from physics.

We know that holomorphic functions or equivalently conformal mappings can be applied to solve the two-dimensional electrostatics, steady temperature distribution, and steady incompressible irrotational fluid flow problems. We can reverse the process and use physical intuition to draw some conclusions about holomorphic functions and conformal mappings. The underlying mathematical methods of the three kinds of physical problems are the same. We choose electrostatics and the fluid flow as the physical models. The main idea is that in the situation of a single sink-source dipole the streamlines and equipotential lines form a global curvilinear coordinate system and from the complex potential function we can get our desired biholomorphic map to the Riemann sphere minus a point or a curve segment.

For the case of compact Riemann surfaces used in the proofs of the theorems of Riemann-Roch and Abel, we do not need any strong conclusions like the biholomorphic property, but the same method of minimizing the Dirichlet integral with prescribed pole-order singularites is used.

We now describe why we expect to have the biholomorphic property in the case of a noncompact simply connected Riemann surface. The property of simply connectedness is used to guarantee the single-valuedness of the conjugate function of the function which we obtain by minimizing the Dirichlet integral.

Suppose we have opposite charges of strength $-q$ and q placed at the points -a and a on the real axis. Let $u(z)$ denote the potential function so that the intensity of the field is the negative of the gradient of $u(z)$. Let $F(z)$ be a holomorphic function whose real part is $u(z)$. The function $F(z)$ is known as the *complex potential* of the field. Let $v(z)$ be the imaginary part of $F(z)$. The field lines are the level lines of $v(z)$. The total field energy is a universal constant times the Dirichlet integral for $u(z)$. In our case of two opposite charges of equal strength the complex potential is given by

$$F = 2q\sqrt{-1}\log\frac{z+a}{z-a}.$$

We can also interpret this as the situation of one source and one sink of equal magnitude in the case of fluid flow. The streamlines are the level lines of $v(z)$. Suppose q equals a constant b times $\frac{1}{a}$ and let a approach 0. Then we get an electric dipole and F becomes $\frac{4b}{z}$. This is precisely the situation we are interested in, namely a harmonic function $u = \operatorname{Re} F$ which has the singularity of the real part of a meromorphic function with a simple pole. In

the case of fluid flow one may have trouble interpreting a sink-source dipole in terms of physical phenomena of the real world.

Now imagine that we have an electric or sink-source dipole on our simply connected Riemann surface. Let us call the level lines of $v(z)$ flow lines (which are the field lines and streamlines respectively in the electrostatics and fluid flow interpretations). Intuitively we expect that all the flow lines $v = c$ are closed except one $v = c_0$ which is along the direction of the axis of the dipole. Let us first look at the special case when our Riemann surface is a bounded domain whose boundary is the boundary of the flow (say in the case of fluid flow). Then the boundary itself is also a flow line and the exceptional flow line $v = c_0$ goes from the dipole to a boundary point ζ_1, where it forks, and then returns from another boundary point ζ_2 to the dipole.

Now consider the mapping from the Riemann surface to $\mathbf{P}_1$ given by

$$w = f(z) = u(z) + \sqrt{-1}\, v(z).$$

The flow lines $v = c$ corresponds to horizontal lines in the w-plane. Since the constant c can take on all values, the whole w-plane is covered by the image of F. The image of the flow line is not a simple horizontal line, because a certain segment L on this horizontal line is the image of two parts of the boundary determined by ζ_1 and ζ_2 and thus is covered twice. So the function $w = f(z)$ maps the domain conformally onto the w-Riemann sphere with a slit L. The w-Riemann sphere minus the slit L is biholomorphic to the open unit disk Δ (or to $\mathbb{C}$ when the slit degenerates to a single point). That is the reason why we expect to get the uniformization theorem from minimizing the Dirichlet problem by a function with singularity equal to the real part of a simple pole. This technique works not only in the case of a bounded domain but also in the case of an abstract open Riemann surface which is simply connected.

§2. *Dirichlet Integrals.*

Let $u(z)$ be a function on domain Ω in a Riemann surface M. We define its *Dirichlet integral* $D[u; \Omega]$ over Ω by

$$D[u;\Omega] = \int_\Omega \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} dxdy,$$

where $z = x + \sqrt{-1}\,y$. When we have another function $v(z)$ we define the Dirichlet inner product

$$D[u, v; \Omega] = \int_\Omega \left\{ \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) \right\} dxdy.$$

Another way to write the Dirichlet integral and inner product is to use the differential operators

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \sqrt{-1}\,\frac{\partial u}{\partial y}\right)$$

and

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \sqrt{-1}\,\frac{\partial u}{\partial y}\right).$$

Let

$$\partial u = \left(\frac{\partial u}{\partial z}\right) dz$$

and

$$\bar{\partial} u = \left(\frac{\partial u}{\partial \bar{z}}\right) d\bar{z}.$$

Then

$$D[u; \Omega] = \int_\Omega \frac{1}{2\sqrt{-1}}\, \partial u \wedge \bar{\partial} u$$

and

$$D[u, v; \Omega] = \int_\Omega \frac{1}{2\sqrt{-1}}\, \partial u \wedge \bar{\partial} v.$$

This means that we can define $D[u; \Omega]$ and $D[u, v; \Omega]$ also when Ω is a Riemann surface. Moreover, since

$$D[u, v; \Omega] = D[u \circ f, v \circ f; \Omega']$$

for any biholomorphic map $f : \Omega' \to \Omega$ and for f equal to the complex-conjugation $z \mapsto \bar{z}$, it follows that the Dirichlet integral of a function is equal to its reflection with respect to a circular arc.

We have the following Green's identity

$$D[u, v; \Omega] + \int_\Omega (\Delta u)v\,dxdy = \int_{\partial \Omega} v\left(u_x dy - u_y dx\right),$$

where the subscripts x and y denote partial differentiation. This is just a consequence of the theorem converting the line integral on the right-hand side to the double integral on the left-hand side.

A consequence of the Green's identity is the following minimizing property of a harmonic function among all functions with prescribed boundary value. Suppose u is a harmonic function on Ω which is continuous on the topological closure $\bar{\Omega}$ of Ω. Suppose w is a function which is continuous and piecewise differentiable on Ω with a convergent Dirichlet integral whose boundary value agrees with that of u. Then $D[u; \Omega] \leq D[w; \Omega]$ and equality holds if and only if $u = w$ on Ω. One gets this by simply plucking in $v = u - w$ in the Green's identity and conclude that $D[u, v; \Omega] = 0$ and

$$D[w; \Omega] = D[u + v; \Omega] = D[u; \Omega] + D[v; \Omega] \geq D[u; \Omega]$$

with equality only when $D[v; \Omega]$ vanishes, which together with the zero boundary value of v means that v is identically zero. Conversely, any function u that is piecewise C^2 on Ω and continuous on $\bar{\Omega}$ is harmonic if it minimizes the Dirichlet integral among all such functions with the same boundary value. This converse follows from the usual derivation of the Euler-Lagrange equation. This simple consequence of Green's identity together with its converse establishes the relation between minimizing functions and harmonic functions. We have to deal with the existence question for the minimizing problem to produce our harmonic function used in the uniformization thereom. On the unit disk we know that the Poisson integral gives us the harmonic function with prescribed boundary values. In general we will extract a convergent subsequence from a minimizing sequence to get our solution as the limit.

§3. *Schwarz Reflection for Harmonic Functions*

In our construction of the minimizing harmonic function we need the Schwarz reflection principle for harmonic functions. Suppose we have a domain Ω symmetric with respect to the reflection about the x-axis. Let u be a harmonic function defined on the intersection of the open upper half-plane H and Ω which is continuous up to $H \cap \partial\Omega$ and vanishes on $H \cap \partial\Omega$. The Schwarz reflection principle asserts that the function on all of Ω defined by $u(z) = -u(\bar{z})$ is harmonic on Ω. This is not just a simple consequence of the Schwarz reflection principle for holomorphic functions which requires that

the holomorphic function on $\Omega \cap H$ be continuous up to $H \cap \partial\Omega$, because the harmonic conjugate of u may not be continuous up to $H \cap \partial\Omega$.

To prove this, without loss of generality we can assume that Ω is the open unit disk Δ and that u is continuous up to the boundary of $\Delta \cap H$. Let

$$P(z, \zeta) = \frac{\zeta\bar{\zeta} - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})}$$

be the Poisson kernel (without the constant factor $\frac{1}{2\pi}$). The reason why the Schwarz reflection principle works is that P satisfies $P(z, \bar{\zeta}) = P(\bar{z}, \zeta)$, which simply says that P is invariant under conjugation (a fact clear from the invariance of the unit circle under conjugation or from P being real). Let

$$w(z) = \frac{1}{2\pi} \int_{\partial\Delta} P(z, \zeta) u(\zeta) d(\arg \zeta)$$

be the harmonic function with the same boundary value as u. To show that u is harmonic, it is the same as showing that w agrees with u. From the symmetry properties $P(z, \bar{\zeta}) = P(\bar{z}, \zeta)$ of P and $u(z) = -u(\bar{z})$ of u it follows that we have the symmetry property $w(z) = -w(\bar{z})$. More precisely,

$$\begin{aligned}
w(\bar{z}) &= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} P\left(\bar{z}, e^{\sqrt{-1}\theta}\right) u\left(e^{\sqrt{-1}\theta}\right) d\theta \\
&= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} P\left(z, e^{-\sqrt{-1}\theta}\right) u\left(e^{\sqrt{-1}\theta}\right) d\theta \\
&= -\frac{1}{2\pi} \int_{\theta=0}^{-2\pi} P\left(z, e^{\sqrt{-1}\theta}\right) u\left(e^{-\sqrt{-1}\theta}\right) d\theta \\
&= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} P\left(z, e^{\sqrt{-1}\theta}\right) u\left(e^{-\sqrt{-1}\theta}\right) d\theta \\
&= -\frac{1}{2\pi} \int_{\theta=0}^{2\pi} P\left(z, e^{\sqrt{-1}\theta}\right) u\left(e^{\sqrt{-1}\theta}\right) d\theta = -w(z).
\end{aligned}$$

Another way to say it is that since P is invariant under conjugation, the result of applying P to u enjoys the same kind of symmetry property of u with respect to conjugation. This means that w vanishes on the real axis and it has the same boundary value as u on the upper half of the unit disk. Hence w agrees with u.

By using a Möbius transformation we conclude that we have the reflection principle also for reflecting with respect to a circular arc instead of the real axis.

The Schwarz reflection principle will be used in the following convergence statement which is essential later when we use compactness arguments to extract a convergent subsequence from a minimizing sequence. Let u_n be a sequence of harmonic functions with finite Dirichlet integral on a domain Ω. Let L be an open circular arc contained in the boundary of Ω. Suppose that each u_n has zero boundary value on L. If $D[u_n; \Omega]$ approaches 0 as $n \to \infty$. then u_n converges to 0 uniformly on compact subsets of $\Omega \cup L$. Let Ω' be the union of Ω and its reflection with respect to L. We use the Schwarz reflection principle to extend each u_n to Ω'. Then $D[u_n; \Omega']$ also converges to 0 as $n \to \infty$. Since any derivative of u_n is harmonic, from the Poisson formula we conclude that any derivative of u_n of *positive* order converges to 0 uniformly on compact subsets of Ω'. Since u_n vanishes at some point of Ω' (say points of L), we conclude that u_n converges to 0 uniformly on compact subsets of Ω'.

§4.*Minimizing Sequence*

The usual way to produce a harmonic function on a smooth bounded domain in $\mathbb{C}$ with prescribed boundary values is to minimize the Dirichlet integral of functions with prescribed boundary values. This does not work in our case of an open Riemann surface without any boundary. If we try to minimize the Dirichlet integral without any boundary condition, we would simply get constant functions. Since eventually we would like to get the sink-source dipole situation, we would like our harmonic function to have the real part of a function with a simple pole. We take an open Riemann surface M and fix a point P_0 in it. Let D_0 be an open coordinate disk centered at P_0 with coordinate z and radius a. Let q be the real part of a meromorphic function on D_0 whose only pole is a simple pole at P_0. We will put more conditions on q later. It is natural that in minimizing the Dirichlet integral we consider only functions u with the property that $u - q$ is smooth on D_0. However, we have a problem when we consider the Dirichlet integral of such functions u over the whole Riemann surface M, because the singularity of u at P_0 would make its global Dirichlet integral diverge. To overcome this difficulty we do the following. For any function h on M we define another function $\hat{h}$ which agrees with $h - q$ on D_0 and agrees with h on $M - D_0$.

We call $D[\hat{h}; M]$ the *normalized Dirichlet integral* of h over M. We seek to minimize $D[\hat{h}; M]$ instead of the infinite global Dirichlet integral $D[h; M]$.

We now consider the variational problem for the functional $D[\hat{h}; M]$. The Euler Lagrange equation is $\Delta(h-q) = 0$ on D_0 and $\Delta h = 0$ on $M - D_0$. Since q is harmonic on D_0, the Euler-Lagrange equation is the same as $\Delta h = 0$. Since the functional $D[\hat{h}; M]$ consists of two integrals over domains of M with boundaries, when we consider the variation problem for the functional $D[\hat{h}; M]$, we look at the variation problem for $D[u; \Omega]$ and add up the two results with $D[u; \Omega] = D[h - q; D_0]$ and $D[u; \Omega] = D[h; M - D_0]$. Let $\frac{\partial}{\partial n}$ mean normal derivative. We have

$$\delta \int_\Omega (u_x^2 + u_y^2) = \int_\Omega 2u_x(\delta u)_x + 2u_y(\delta u)_y$$

$$= 2D[u, \delta u; \Omega] = -2 \int_\Omega (\Delta u)(\delta u) + 2 \int_{\partial\Omega} \delta u \frac{\partial u}{\partial n}$$

by Green's formula. When we choose δu with support in the interior of Ω we conclude that $\Delta u = 0$ for a critical function u. Hence

$$\delta \int_\Omega (u_x^2 + u_y^2) = 2 \int_{\partial\Omega} \delta u \frac{\partial u}{\partial n}$$

and

$$\delta\, D[\hat{h}; M] = 2 \int_{\partial D_0} \delta(h - q)\frac{\partial}{\partial n}(h - q) + 2 \int_{\partial(M-D_0)} \delta h \frac{\partial h}{\partial n}$$

$$= 2 \int_{\partial D_0} \delta h \frac{\partial}{\partial n}(h - q) + 2 \int_{\partial(M-D_0)} \delta h \frac{\partial h}{\partial n}$$

$$= -2 \int_{\partial D_0} \delta h \frac{\partial q}{\partial n}.$$

So a necessary condition for the existence of a critical function u for the functional $D[\hat{h}; M]$ is the vanishing of the normal derivative of q at ∂D_0 and we have to put this additional condition on q. We determine the form of q by using the method of undetermined coefficients. By using the Laurent series of the meromorphic function whose real part is q, we write q in the form

$$q = \text{Re}\left(\frac{1}{z} + \sum_{\nu=1}^{\infty} c_\nu z^\nu\right).$$

We drop the constant term because it does not affect the condition of the vanishing of the normal derivative of q at ∂D_0. To compute the normal derivative of q we use polar coordinates $z = re^{\sqrt{-1}\,\theta}$. We have

$$\frac{\partial}{\partial n}\,q = \frac{-1}{r^2}\cos\theta + \sum_{\nu=1}^{\infty} \nu\, c_\nu r^{\nu-1}\cos\nu\theta$$

which vanishes identically on $r = a$ for all values of θ if and only if $c_\nu = 0$ for $\nu > 1$ and $c_1 = \frac{1}{a^2}$. So from now on we use the function

$$q = \mathrm{Re}\left(\frac{1}{z} + \frac{z}{a^2}\right) = \frac{x}{x^2 + y^2} + \frac{x}{a^2}.$$

We call a function h a *comparison function* if h is continuous and piece-wise differentiable on $M - P_0$ and $h - q$ is continuous at P_0 and the Dirichlet integral $D[\hat{h}, M]$ is finite. We want to minimize $D[\hat{h}, M]$ among all comparison functions h. Let μ be the infimum of $D[\hat{h}, M]$. Choose a sequence h_n so that $D[\hat{h}_n, M]$ approaches μ as $n \to \infty$. Let us call such a sequence h_n a *minimizing sequence*. The main result at this point is the infimum μ is actually achieved by some comparison function u. We are going to get u by constructing convergent minimizing sequence from any given minimizing sequence so that u is harmonic on $M - P_0$.

First we want to say that when $D[\hat{h}_n, M]$ approaches μ as $n \to \infty$, the sequence h_n is a Cauchy sequence with respect to the norm $D[\hat{h}_n; M]$. This is done by using a triangle inequality. To motivate such a triangle inequality, let us consider the following. If the sequence h_n converges to our desired limit function u as we expect it would, we can conclude that the sequence h_n is Cauchy by using the triangle inequality

$$\sqrt{D[\hat{h}_n - \hat{h}_m; M]} \leq \sqrt{D[\hat{h}_n - u; M]} + \sqrt{D[\hat{h}_m - u; M]}.$$

However, we do not have our limit function u yet. We rewrite our triangle inequality without explicitly using the function u which is not yet existent. Let us for the time being assume that we have u and try to see what the above triangle inequality looks like without explicitly using u. From $h = u + (h - u)$ we get

$$D[\hat{h}; M] = D[\hat{u}; M] + 2\,D[\hat{u}, h - u; M] + D[h - u; M].$$

Since u is critical for the functional $D[\hat{u}; M]$, from

$$D[\hat{u}; M] \le D[\hat{u} + \lambda f; M] = D[\hat{u}; M] + 2\,\lambda D[\hat{u}, f; M] + \lambda^2 D[f; M]$$

for any f and λ we conclude that $D[\hat{u}, f; M]$ must vanish for any f. Hence

$$D[\hat{h}; M] = D[\hat{u}; M] + 2\,D[\hat{u}, h - u; M] + D[h - u; M] = \mu + D[h - u; M]$$

and $D[h - u; M] = D[\hat{h}; M] - \mu$. Thus the triangle inequality for the Dirichlet integral should be formulated as following. For any two comparison functions g and h we have

$$\sqrt{D[g - h; M]} \le \sqrt{D[\hat{g}; M] - \mu} + \sqrt{D[\hat{h}; M] - \mu}.$$

For the left-hand side we did not write $\hat{g} - \hat{h}$ because it is the same as $g - h$. The proof of this triangle inequality is the standard argument of using the positive definiteness criterion of a quadratic form. Since

$$D[\lambda f + \hat{h}; M] = \lambda^2 D[f, M] + 2\lambda D[f, \hat{h}; M] + D[\hat{h}; M] \ge \mu,$$

it follows that

$$|D[f, \hat{h}; M]| \le \sqrt{D[f; M](D[\hat{h}; M] - \mu)}.$$

We have

$$|D[f, \hat{g}; M]| \le \sqrt{D[f; M](D[\hat{g}; M] - \mu)}$$

by replacing h by g. Now letting $f = g - h$ we get

$$D[g - h; M] = D[g - h, \hat{g}; M] - D[g - h, \hat{h}; M]$$

$$\le \sqrt{D[g - h; M](D[\hat{g}; M] - \mu)} + \sqrt{D[g - h; M](D[\hat{h}; M] - \mu)}$$

and the triangle inequality follows. From this triangle inequality we conclude that for any minimizing sequence h_n we have $D[h_n - h_m; M]$ goes to 0 as n, m goes to infinity.

We cover M with a countable number of open coordinate disks D_ν so that the family $\{D_\nu\}_{\nu \ge 0}$ is locally finite and the first member D_0 is the coordinate disk centered at P_0 which we have chosen earlier. We also assume that D_ν is not contained in $D_0 \cup \cdots \cup D_{\nu-1}$ to avoid redundancy. We also assume that

the first two coordinate disks D_0, D_1 are concentric with the same coordinate so that D_0 is relatively compact in D_1 and the other $D'_\nu s$ with $\nu \geq 2$ do not intersect D_0 (of course some of them intersect D_1). We now use the Perron argument of replacing on a disk a function by a harmonic function with the same boundary value. Suppose we have a test function h. We replace h on D_1 by a function g so that $g - q$ is harmonic on D_1 and g and h agree on the boundary of D_1. We know that $D[g - q; D_1] \leq D[h - q; D_1]$. We claim that $D[\hat{g}; \Omega] \leq D[\hat{h}; \Omega]$. In other words

$$D[g - q; D_0] + D[g; D_1 - D_0] \leq D[h - q; D_0] + D[h; D_1 - D_0].$$

From $D[g - q; D_1] \leq D[h - q; D_1]$ we have

$$D[g - q; D_0] + D[g - q; D_1 - D_0] \leq D[h - q; D_0] + D[h - q; D_1 - D_0],$$

i.e.,

$$
\begin{aligned}
&D[g - q; D_0] + D[g; D_1 - D_0] - 2D[g, q; D_1 - D_0] + D[q; D_1 - D_0] \\
&\leq D[h - q; D_0] + D[h; D_1 - D_0] - 2D[h, q; D_1 - D_0] + D[q; D_1 - D_0].
\end{aligned}
$$

It suffices to show that $D[g - h, q; D_1 - D_0] = 0$. The Green's identity says that

$$D[g - h, q; D_1 - D_0] + \int_{D_1 - D_0} (\Delta q)(g - h)\mathrm{dxdy} = \int_{\partial D_1 - \partial D_0} (g - h)\frac{\partial q}{\partial n}ds,$$

where $\frac{\partial q}{\partial n}$ is the normal derivative of q. Now since $g - h$ vanishes on ∂D_1 and $\frac{\partial q}{\partial n}$ vanishes on ∂D_0, the boundary integral is zero. This the reason why we earlier chose q so that $\frac{\partial q}{\partial n}$ vanishes on ∂D_0. What we want follows from $\Delta q = 0$. The claim is verified.

Since we know that on a disk a harmonic function minimizes the Dirichlet integral among all functions with prescribed boundary value, it follows that this process enables us to choose a minimizing sequence h_n whose every member is harmonic on D_1. We can replace each $h_n - q$ by $(h_n - q) - (h_n - q)(P_0)$ and assume that $(h_n - q)(P_0) = 0$. Since $D[h_n - h_m; M]$ goes to 0 as n, m goes to infinity, the function $\frac{\partial}{\partial \bar{z}}(h_n - h_m)$ is holomorphic on D_1 whose L^2 norm goes to zero as n, m goes to infinity. Since the function $h_n - h_m$ is real and is normalized to be zero at the point P_0, by integrating along a line segment emanating from P_0 we conclude that $h_n - h_m$ converges

uniformly on compact subsets of D_1 to 0. Thus $h_n - q$ converges uniformly on compact subsets to a harmonic function on D_1. This argument for the convergence of $h_n - q$ depends on the anchoring of the value of the function $h_n - q$ at P_0.

An obvious thing to do is to try to this procedure to any D_ν for $\nu \geq 2$ and get the convergence of h_n to some harmonic function on $M - P_0$. The major difficulty is that we do not have the anchoring of the value of the function h_n at the center of D_2 in the next step. For this anchoring all we need is the assumption that the value of the function h_n converges. At a point which is on the boundary of D_2 but in the interior of D_1 we know that the value of of the function h_n approaches to a limit, though we cannot do this for a point which is interior to both D_1 and D_2 due to the fact that the function h_n is being altered in the interior of D_2 by the Poisson kernel. To be able to use the anchoring at the boundary, we need the Schwarz reflection principle for harmonic functions so that after the function is being reflected with respect to the boundary, the boundary point becomes an interior point. Another difficulty is that when we apply the process of altering the function by using the Poisson kernel on some D_ν we may destroy the harmonicity which we have achieved for some D_λ with $\lambda < \nu$ if D_λ intersects the boundary of D_ν. However, what we are interested in is not that each h_n is harmonic, but the limit of h_n as $n \to \infty$ is harmonic. We get the harmonicity of the limit again by using the Schwarz reflection principle for harmonic functions. Let us argue intuitively first and then we do it rigorously. Suppose we replace h_n by g_n on the disk D_ν so that g_n is harmonic on D_ν ($\nu \geq 2$). Then the normalized Dirichlet integral of g_n is no more than that of h_n. From the triangle inequality we know that the normalized Dirichlet integral of $g_n - h_n$ approaches zero, from which we will conclude that the limit function $\lim_{n\to\infty} h_n$ and the limit function $\lim_{n\to\infty} g_n$ differ at most by a constant on D_ν. Since g_n is harmonic on D_ν, it follows that $\lim_{n\to\infty} g_n$ is harmonic on D_ν and $\lim_{n\to\infty} h_n$ is harmonic on D_ν. Now we do the process rigorously.

For each $\lambda \geq 2$, we successfully replace $h_n|D_\lambda$ by a function with the same boundary value and harmonic on D_λ. We claim that the new sequence converges uniformly on compact subset of $M - \{P_0\}$ to a harmonic function. Since the family $\{D_0\}$ is locally finite, for any compact subset K of M the member D_ν is disjoint from K for ν sufficiently large. To prove the claim, we need only verify that for any $k \geq 2$, if we successfully replace $h_n|D_\lambda$ by a function with the same boundary value and harmonic on D_λ for $2 \leq \lambda < k$,

then the new sequence converges uniformly on compact subset of

$$\bigcup_{\lambda < k} D_\lambda - \{P_0\}$$

to a function which is harmonic on

$$\bigcup_{\lambda < k} D_\lambda - \{P_0\}.$$

We are going to verify by induction on k.

The case of $k = 2$ has been verified. Suppose we have done this for k and we want to do it when k is replaced by $k + 1$. Without loss of generality we assume that h_n is already the sequence obtained by successfully replacing $h_n|D_\lambda$ by a function with the same boundary value and harmonic on D_λ for $2 \leq \lambda < k$. Our induction hypothesis says that h_n converges uniformly on compact subset of

$$\bigcup_{\lambda < k} D_\lambda - \{P_0\}$$

to a function u_k which is harmonic on

$$\bigcup_{\lambda < k} D_\lambda - \{P_0\}.$$

We now do the new replacement on D_k. So we replace $h_n|D_k$ by a function with the same boundary value and harmonic on D_k and obtain a minimizing sequence g_n. Since g_n is a minimizing sequence, by the triangle inequality we know that

$$D[g_n - g_m; M] \to 0$$

as $m, n \to \infty$. So the positive-order derivatives of the harmonic functions $g_n|D_k$ converges uniformly on compact subsets of D_k. We have to show that g_n converges uniformly on compact subset of

$$\bigcup_{\lambda \leq k} D_\lambda - \{P_0\}$$

to a harmonic function. We have to worry about the harmonicity of the limit at points of the set

$$\bigcup_{\lambda < k} (D_\lambda \cap \partial D_k)$$

which without loss of generality we can assume to be nonempty.

Fix $\lambda < k$ so that $D_\lambda \cap \partial D_k$ is nonempty. Let

$$G_\lambda = \left(D_\lambda - \bigcup_{\lambda < \mu < k} \partial D_\mu \right) \cap D_k,$$

$$C_\lambda = \left(D_\lambda - \bigcup_{\lambda < \mu < k} \partial D_\mu \right) \cap \partial D_k.$$

Then C_λ is a disjoint union of circular arcs. Let G'_λ be the reflection of G_λ with respect to ∂D_k and let

$$\tilde{G}_\lambda = G_\lambda \cup C_\lambda \cup G'_\lambda.$$

The function $g_n - h_n$ is harmonic on G_λ and vanishes on C_λ and so can be extended by the Schwarz reflection principle to a function f_n which is harmonic on $\tilde{G}_\lambda$. Since both g_n and h_n are minimizing sequences, by the triangle inequality we know that $D[g_n - h_n; G_\lambda]$ converges to 0 as $n \to \infty$. Since

$$D[f_n, G'_\lambda] = D[f_n, G_\lambda] = D[g_n - h_n; G_\lambda] \to 0$$

as $n \to \infty$, it follows that $D[f_n; \tilde{G}_\lambda]$ converges to 0 as $n \to \infty$. Because f_n is harmonic on $\tilde{G}_\lambda$ and vanishes on C_λ, we know that f_n converges uniformly on compact subsets of $\tilde{G}_\lambda$ to zero. (Note that, all we need in our argument is that $f_n|G_\lambda$ converges to 0 as $n \to \infty$ and we can avoid using the reflection here and directly quote the result in §3 where the Schwarz reflection for harmonic functions is given.)

Since on $G_\lambda \subset D_\lambda$ the sequence h_n approaches u, it follows that $g_n = h_n + f_n$ approaches the function u on G_λ. Because g_n is a minimizing sequence, we know that $D[g_n - g_m; D_k] \to 0$ as $m, n \to \infty$ and, as a result, the positive-order derivatives of the harmonic functions $g_n|D_k$ converges uniformly on compact subsets of D_k. Since $g_n|G_\lambda$ converges to $u|G_\lambda$ as $n \to \infty$ and since G_λ is a nonempty subset of D_k, it follows that $g_n|D_k$ converges uniformly on compact subsets of D_k to a harmonic function $u^\#$ on D_k, which agrees with u on G_λ.

Since both u and $u^\#$ are harmonic on

$$\left(\bigcup_{\lambda < k} D_\lambda \right) \cap D_k$$

and they agree on the nonempty subset G_λ, it follows from the identity theorem for harmonic functions that u and $u^\#$ agree on

$$\left(\bigcup_{\lambda < k} D_\lambda \right) \cap D_k.$$

Let u_{k+1} be the function on

$$\bigcup_{\lambda < k+1} D_\lambda - \{P_0\}$$

which agrees with u_k on

$$\bigcup_{\lambda < k} D_\lambda - \{P_0\}$$

and agrees with $u^\#$ on D_k. Then g_n converges on compact subsets of

$$\bigcup_{\lambda < k} D_\lambda - \{P_0\}$$

to the harmonic function u_{k+1} and the proof by induction is complete.

By the identity theorem for harmonic functions we know that the limit u on $\bigcup_{\lambda \leq k} D_\lambda$ is same for each step. Since the covering $\{D_\lambda\}$ is locally finite, by the usual diagonalizing process we can actually choose one h_{n_k} from the kth step to get a minimizing sequence which converges to u uniformly on compact subsets. Thus we have the following.

Final Conclusion. Given a coordinate disk D_0 of radius a centered at a point P of a Riemann surface M. Let

$$q = \text{Re}\left(\frac{1}{z} + \frac{z}{a^2} \right) = \frac{x}{x^2 + y^2} + \frac{x}{a^2},$$

where $z = x + \sqrt{-1}\, y$ is the coordinate of D_0 (so that the normal derivative of q vanishes at the boundary of D_0). Then there exists a harmonic function u on $M - \{P\}$ such that $u - q$ is harmonic on D_0. The function u can be constructed by minimizing the (modified) Dirichlet integral

$$\int_{M - D_0} |\text{grad}\, u|^2 + \int_{D_0} |\text{grad}\, (u - q)|^2.$$

§5. *Independence of Choice of Covering.* For a given point P and a local coordinate at P, this minimizing function u is actually unique and is independent of the choice of $\{D_\lambda\}_{\lambda \geq 1}$ and the radius a of D_0 and the choice of $\{h_\lambda\}$. First we look at the question about the choice of $\{h_\lambda\}$. This is a consequence of the observation that u minimizes $D[\hat{u}; M]$ if and only if $D[\hat{u}, h; M]$ vanishes for any piecewise smooth function h with $D[h; M]$ finite. The "only if" part is obtained by looking at the criterion of the positive-definiteness of the quadratic form $D[\hat{u} + \lambda h; M] - \mu$ in λ and using $D[\hat{u}; M] = \mu$. More precisely,

$$D[\hat{u}; M] + 2\, D[\hat{u}, h, M]\lambda + \lambda^2 D[h; M] \geq \mu = D[\hat{u}; M]$$

implies that

$$2\, D[\hat{u}, h, M]\lambda + \lambda^2 D[h; M]$$

is nonnegative for all real λ and so $D[\hat{u}, h, M]$ must vanish. Conversely if some function v (with normalization $(v - q)\,(P_0) = 0$) satisfies the condition, then

$$\mu = D[\hat{u}; M] = D[\hat{v} + (u - v); M] = D[\hat{v}; M] + D[u - v; M] \geq D[\hat{v}; M] \geq \mu$$

and v must be equal to u identically and is a minimizing function.

As for the choice of $\{D_\lambda\}$ what matters is the disk D_0, because when we have a different covering with the same D_0 we can apply the criterion just observed above to get the uniqueness. Now suppose we have another disk D_0^* with radius $a^* > a$ which is concentric with D_0. Then we have another function q^* with singularity. Denote by u^* the function which agrees with $u - q^*$ on D_0^* and agrees with u on $M - D_0^*$. Again by the criterion observed above it suffices to show that $D[u^*, h; M] = 0$ for any piecewise smooth function h with finite Dirichlet integral. This means that we have to prove $D[\hat{u} - u^*, h; M] = 0$ for any piecewise smooth function h with finite Dirichlet integral. The function $\hat{u} - u^*$ equals $(u - q) - (u - q^*) = q^* - q$ on D_0 and equals $u - (u - q^*) = q^*$ on $D_0^* - D_0$ and equals 0 on $M - D_0^*$. Thus

$$D[\hat{u} - u^*, h; M] = D[q^* - q, h; D_0] + D[q^*, h; D_0^* - D_0].$$

By Green's identity

$$D[q^* - q, h; D_0] + \int_{D_0} (\Delta(q^* - q))h\,dxdy = \int_{\partial D_0} \frac{\partial}{\partial n}(q^* - q)h\ ds,$$

$$D[q^*, h; D_0^* - D_0] + \int_{D_0^* - D_0} (\Delta q^*) h \, dx dy = \int_{\partial D_0^* - \partial D_0} \frac{\partial q^*}{\partial n} h \, ds.$$

Hence from the harmonicity of $q^* - q$ on D_0 and the harmonicity of q^* on $D_0^* - D_0$ we have

$$D[q^* - q, h; D_0] + D[q^*, h; D_0^* - D_0] = \int_{\partial D_0} \frac{\partial}{\partial n}(q^* - q) h \, ds + \int_{\partial D_0^* - \partial D_0} \frac{\partial q^*}{\partial n} h \, ds$$

$$= -\int_{\partial D_0} \frac{\partial q}{\partial n} h \, ds + \int_{\partial D_0^*} \frac{\partial q^*}{\partial n} h \, ds = 0.$$