Theorem of Hardy-Littlewood on Infinite Number of Zeroes of Riemann Zeta Function on Critical Line

Associate Functions of Riemann Zeta Function and Their "Real-Coefficient" Property. We recall the functional equation of the Riemann zeta function.

$$\xi(s) = \xi(1-s),$$

where

$$\xi(s) = \frac{s(1-s)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Define

$$\chi(x) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Then the functional equation of Riemann zeta function can be rewritten as

$$\chi(x) = \frac{\zeta(s)}{\zeta(1-s)}.$$

By the duplication formula for the Gamma function and the relation between the Gamma function and sine function, χ can be written also as

$$\chi(x) = 2^s\pi^{s-1}\sin\frac{s\pi}{2}\Gamma(1-s) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = 2^{s-1}\pi^s\frac{\sec\frac{\pi s}{2}}{\Gamma(s)}.$$

On the critical line $\operatorname{Re} s = \frac{1}{2}$ the zeroes of $\zeta(s)$ are the same as the zeroes of $\xi(s)$. In order to more conveniently locate the zeroes of $\zeta(s)$ we focus on $\xi(s)$ and make a change of variables by letting

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right).$$

so that

$$\Xi(z) = \Xi(-z)$$

and the investigation of the zeroes of $\zeta(s)$ on the critical line becomes the investigation of the *real* zeroes of $\Xi(z)$. The functional equation now becomes the *evenness* of the function $\Xi(z)$.

We now look at the "real-coefficient" property of the associated functions of $\zeta(s)$ which is also called the real character of such functions. We have the real character of the functions ξ and Ξ given by

$$\overline{\xi(s)} = \frac{\bar{s}(\bar{s}-1)}{2}\pi^{-\frac{\bar{s}}{2}}\Gamma\left(\frac{\bar{s}}{2}\right)\xi(\bar{s}) = \xi(\bar{z}),$$
$$\overline{\Xi(z)} = \overline{\xi\left(\frac{1}{2}+iz\right)} = \xi\left(\frac{1}{2}-i\bar{z}\right) = \Xi\left(-\bar{z}\right) = \Xi\left(\bar{z}\right).$$

Reduction of Problem to Number of Real Zeroes of Some Real-Valued Function on Real Line. We will use the important information that

$$\left|\chi\left(\frac{1}{2}+it\right)\right| = 1.$$

It follows from

$$\chi(x) = \frac{\zeta(s)}{\zeta(1-s)} \implies \chi(s)\chi(1-s) = 1 \implies \chi\left(\frac{1}{2}+it\right)\chi\left(\frac{1}{2}-it\right) = 1$$

and

$$\overline{\chi(s)} = \chi(\bar{s}) \implies \left|\chi\left(\frac{1}{2}+it\right)\right| = 1.$$

Let

$$\vartheta = \vartheta(t) = -\frac{1}{2}\arg\chi\left(\frac{1}{2}+it\right)$$

so that

$$\chi\left(\frac{1}{2}+it\right) = e^{-2i\vartheta}.$$

Let

$$Z(t) = e^{i\vartheta}\zeta\left(\frac{1}{2}+it\right) = \frac{\zeta\left(\frac{1}{2}+it\right)}{\chi\left(\frac{1}{2}+it\right)^{\frac{1}{2}}}.$$

The zeroes of the zeta function on the critical line is the same as the zeroes of $Z(t)$ on the real line. The function $Z(t)$ will be shown to be real-valued on the real line. The key point of the proof of the theorem of Hardy-Littlewood is that from

$$\left|\chi\left(\frac{1}{2}+it\right)\right| = 1$$

we have different growth rates of $Z(t)$ and $|Z(t)|$. The difference of the growth rate of

$$\int_{t=T}^{2T} Z(t)dt$$

and

$$\int_{t=T}^{2T} |Z(t)|\, dt$$

will show that the real-valued function $Z(t)$ must change signs an infinite number of times on the real axis, implying that $Z(t)$ has an infinite number of real zeroes.

Difference in Growth Rate of an Integral with Integrand Replaced by its Absolute Value. The difference in the growth rates of

$$\int_{t=T}^{2T} Z(t)dt$$

and

$$\int_{t=T}^{2T} |Z(t)|\, dt$$

hinges on cancelations from the change of arguments in

$$\chi\left(\frac{1}{2}+it\right)^{-\frac{1}{2}}.$$

Now

$$\chi(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$$

and

$$\chi\left(\frac{1}{2}+it\right)^{-\frac{1}{2}} = \pi^{-\frac{1}{2}it}\frac{\Gamma\left(\frac{1}{4}-\frac{1}{2}it\right)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)^{-\frac{1}{2}}} = \pi^{-\frac{1}{2}it}\frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{4}-\frac{1}{2}it\right)^{\frac{1}{2}}}$$

$$= \pi^{-\frac{1}{2}it}\frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{4}-\frac{1}{2}it\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)^{\frac{1}{2}}} = \pi^{-\frac{it}{2}}\frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\right|},$$

it follows that

$$Z(t) = \frac{\zeta\left(\frac{1}{2}+it\right)}{\chi\left(\frac{1}{2}+it\right)^{\frac{1}{2}}}$$

$$= \pi^{-\frac{it}{2}} \frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\right|} \zeta\left(\frac{1}{2}+it\right)$$
$$= -2\pi^{\frac{1}{4}} \frac{\Xi(t)}{\left(t^2+\frac{1}{4}\right)\left|\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\right|},$$

which implies, in particular, from $\overline{\Xi(t)} = \Xi(t)$ that $Z(t)$ is real-valued on the real line. From

$$\left|\chi\left(\frac{1}{2}+it\right)\right| = 1$$

we have

$$|Z(t)| = \left|\frac{\zeta\left(\frac{1}{2}+it\right)}{\chi\left(\frac{1}{2}+it\right)^{\frac{1}{2}}}\right| = \left|\zeta\left(\frac{1}{2}+it\right)\right|.$$

We now compare

$$\int_T^{2T} Z(t)dt$$

and

$$\int_T^{2T} |Z(t)|\, dt.$$

Use

$$Z(t) = \frac{\zeta\left(\frac{1}{2}+it\right)}{\chi\left(\frac{1}{2}+it\right)^{\frac{1}{2}}}$$

to get

$$\int_{\frac{1}{2}+iT}^{\frac{1}{2}+2iT} \chi(s)^{-\frac{1}{2}}\, \zeta(s)\, ds = i \int_T^{2T} Z(t)dt,$$

where the i from the right-hand side comes from iT and $2iT$.

Use of Stirling's Formula. Recall that from the use of Stirling's formula and the form

$$\chi(s) = 2^{s-1}\pi^s \frac{\sec\frac{\pi s}{2}}{\Gamma(s)}$$

of the function $\chi(s)$, we have the following growth estimate for $\chi(s)$.

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-\frac{1}{2}} e^{i\left(t+\frac{\pi}{4}\right)} \left(1+O\left(\frac{1}{t}\right)\right)$$

and

$$\chi(s)^{-\frac{1}{2}} = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}\sigma-\frac{1}{4}+\frac{1}{2}it} e^{-\frac{it}{2}-\frac{i\pi}{8}} \left(1 + O\left(\frac{1}{t}\right)\right).$$

Growth Rate of Riemann Zeta Function on Vertical Lines. Recall the following growth rate of the Riemann zeta function on vertical lines.

$$\zeta(\sigma+it) = O\left(|t|^{\mu(\sigma)+\epsilon}\right)$$

with

$$\begin{cases} \mu(\sigma) = 0 \text{ if } \sigma > 1, \\ \mu(\sigma) = \frac{1}{2} - \sigma \text{ if } \sigma < 0, \\ \mu(\sigma) = \frac{1}{2}(1-\sigma) \text{ if } 0 \leq \sigma \leq 1. \end{cases}$$

Thus

$$\chi(s)^{-\frac{1}{2}}\zeta(s) = O\left(t^{\frac{\sigma}{2}-\frac{1}{4}}t^{\frac{1}{2}-\frac{\sigma}{2}+\epsilon}\right) = O\left(t^{\frac{1}{4}+\epsilon}\right) \text{ for}$$

and

$$\chi(s)^{-\frac{1}{2}}\zeta(s) = O\left(t^{\frac{\sigma}{2}-\frac{1}{4}+\epsilon}\right) = O\left(t^{\frac{3}{8}+\epsilon}\right) \text{ for } 1 < \sigma \leq \frac{5}{4}.$$

Shifting Integration to Critical Line of Abscissa Greater Than Abscissa of Convergence for Riemann Zeta Function. One key point is to shift the path of integration to the critical line with abscissa $\frac{5}{4}$ so that the Dirichlet series representation of $\zeta(s)$ can be used whose terms provide a great deal of oscillation for cancellation when the variable point goes up the vertical line.

For such a shifting of the path of integration, we get two additional terms from the integration of $\chi(s)^{-\frac{1}{2}}\zeta(s)$ with respect to σ from $\sigma = \frac{1}{2}$ to $\sigma = \frac{5}{4}$ on $t = T$ and on $t = 2T$. From the above growth estimates of $\chi(s)^{-\frac{1}{2}}\zeta(s)$ for $\frac{1}{2} \leq \sigma \leq 1$ the error estimate for these two additional terms is $O\left(T^{\frac{3}{8}+\epsilon}\right)$.

We now handle the integration over the critical line of abscissa $\frac{5}{4}$. Integrating $\chi(s)^{-\frac{1}{2}}\zeta(s)$ along $\sigma = \frac{5}{4}$ from $t = T$ to $2T$ gives

$$\int_{t=T}^{2T} \left(\frac{t}{2\pi}\right)^{\frac{3}{8}+\frac{1}{2}it} e^{-\frac{it}{2}-\frac{i\pi}{8}} \left(1 + O\left(\frac{1}{t}\right)\right) \zeta\left(\frac{5}{4}+it\right) idt$$

with $O\left(\frac{1}{t}\right)$ contributing

$$\int_{t=T}^{2T} O\left(t^{-\frac{5}{8}}\right) dt = O\left(T^{\frac{5}{8}}\right)$$

and

$$\begin{aligned}\int_{t=T}^{2T} \left(\frac{t}{2\pi}\right)^{\frac{3}{8}+\frac{1}{2}it} & e^{-\frac{it}{2}-\frac{i\pi}{8}} \zeta\left(\frac{5}{4}+it\right) idt \\ = \sum_{n=1}^{\infty} n^{-\frac{5}{4}} \int_{t=T}^{2T} & \left(\frac{t}{2\pi}\right)^{\frac{3}{8}+\frac{1}{2}it} e^{-\frac{it}{2}-it\log n} dt.\end{aligned}$$

Write

$$\left(\frac{t}{2\pi}\right)^{\frac{3}{8}+\frac{1}{2}it} e^{-\frac{it}{2}-it\log n} = \left(\frac{t}{2\pi}\right)^{\frac{3}{8}} e^{i\left(\frac{t}{2}\log\frac{t}{2\pi}-\frac{t}{2}-t\log n\right)}.$$

Use of Oscillation Lemma. We now apply the "Oscillation Lemma Involving Second Derivative" to

$$G(t) = \left(\frac{t}{2\pi}\right)^{\frac{3}{8}}$$

and

$$F(t) = \frac{t}{2}\log\frac{t}{2\pi} - \frac{1}{2}t - t\log n.$$

We have

$$F''(t) = \frac{d^2}{dt^2}\left(\frac{t}{2}\log\frac{t}{2\pi} - \frac{1}{2}t - t\log n\right) = \frac{1}{2t},$$

which is $\geq \frac{1}{4T}$ on $T \leq t \leq 2T$. From

$$F'(t) = \frac{d}{dt}\left(\frac{t}{2}\log\frac{t}{2\pi} - \frac{1}{2}t - t\log n\right) = \frac{1}{2}\log\frac{t}{2\pi} - \log n$$

and

$$\begin{aligned}G'F' - G\,F'' &= \frac{3}{16\pi}\left(\frac{t}{2\pi}\right)^{-\frac{5}{8}}\left(\frac{1}{2}\log\frac{t}{2\pi} - \log n\right) - \left(\frac{t}{2\pi}\right)^{\frac{3}{8}}\frac{1}{2t} \\ &= t^{-\frac{5}{8}}\left(\frac{3}{16\pi}\left(\frac{1}{2\pi}\right)^{-\frac{5}{8}}\left(\frac{1}{2}\log\frac{t}{2\pi} - \log n\right) - \left(\frac{1}{2\pi}\right)^{\frac{3}{8}}\frac{1}{2}\right)\end{aligned}$$

which is positive when $T \leq t \leq 2T$ and T is sufficiently large, it follows that

$$\frac{d}{dt}\left(\frac{G(t)}{F'(t)}\right) = \frac{G'(t)F'(t) - G(t)\,F''(t)}{F'(t)^2} > 0$$

and the function

$$\frac{G(t)}{F'(t)}$$

is increasing for $T \leq t \leq 2T$ and T is sufficiently large. Since

$$G(t) = O\left(T^{\frac{3}{8}}\right)$$

for $T \leq t \leq 2T$, by the "Oscillation Lemma Involving Second Derivative" we conclude that

$$\int_{t=T}^{2T}\left(\frac{t}{2\pi}\right)^{\frac{3}{8}+\frac{1}{2}it} e^{-\frac{it}{2}-it\log n}dt = \int_{t=T}^{2T} G(t)e^{iF(t)}dt = O\left(T^{\frac{7}{8}}\right)$$

uniformly in n and

$$\int_{t=T}^{2T} Z(t)dt = O\left(T^{\frac{7}{8}}\right).$$

Growth Rate of Integral of Absolute Value of Integrand. We now compute the growth order of

$$\int_{t=T}^{2T} |Z(t)|\,dt.$$

Because

$$\left|\chi\left(\frac{1}{2}+it\right)\right| = 1$$

and

$$|Z(t)| = \left|\frac{\zeta\left(\frac{1}{2}+it\right)}{\chi\left(\frac{1}{2}+it\right)^{\frac{1}{2}}}\right| = \left|\zeta\left(\frac{1}{2}+it\right)\right|,$$

the introduction of the absolute value for the integrand gets rid of the contribution from $\chi(s)$ to the estimate of the integral. We will also shift the path of integration to a vertical line with abscissa greater than the abscissa of convergence of the Dirichlet series of the Riemann zeta function. For this integral estimate we will use the vertical line with abscissa 2. As we see below, the dominant term for the estimate will turn out to be from the constant term 1 in the Dirichlet series expansion

$$\zeta(s) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^s}$$

of the Riemann zeta function. We have

$$\int_{t=T}^{2T} |Z(t)|\, dt = \int_{t=T}^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right| dt \geq \left| \int_{t=T}^{2T} \zeta\left(\frac{1}{2} + it\right) dt \right|$$

and

$$\begin{aligned} i \int_{t=T}^{2T} \zeta\left(\frac{1}{2} + it\right) dt &= \int_{s=\frac{1}{2}+iT}^{\frac{1}{2}+2iT} \zeta(s) ds \\ = \int_{s=\frac{1}{2}+iT}^{2+iT} \zeta(s) ds &+ \int_{s=2+iT}^{2+2iT} \zeta(s) ds + \int_{s=2+2iT}^{2+iT} \zeta(s) ds \\ = \left[s - \sum_{n=2}^{\infty} \frac{1}{n^s \log n} \right]_{s=2+iT}^{2+2iT} &+ \int_{\sigma=\frac{1}{2}}^{2} O\left(T^{\frac{1}{2}}\right) d\sigma \\ &= iT + O\left(T^{\frac{1}{2}}\right), \end{aligned}$$

where the term

$$\left[s - \sum_{n=2}^{\infty} \frac{1}{n^s \log n} \right]_{s=2+it}^{2+2iT}$$

is obtained by integrating ζ directly from its definition as an infinite series and the term

$$\int_{\sigma=\frac{1}{2}}^{2} O\left(T^{\frac{1}{2}}\right) d\sigma$$

is from Lindelöf's result. Hence

$$\int_{t=T}^{2T} |Z(t)|\, dt > AT$$

for some positive number A independent of T. This concludes the proof of the theorem of Hardy-Littlewood on the infinite number of zeroes of the zeta function on the critical line.