Lindelöf Hypothesis

Define
$$\mu(\sigma) = \inf\left\{ a \in \mathbf{R} \,\middle|\, \zeta(\sigma + it) = O\left(|t|^{a}\right) \right\}.$$

We know from the defining series of the Riemann zeta function, the functional equation, and Stirling's formula that

(†)
$$\begin{cases} \mu(\sigma) = 0 \text{ if } \sigma > 1, \\[2mm] \mu(\sigma) = \frac{1}{2} - \sigma \text{ if } \sigma < 0. \end{cases}$$

By interpolation from the theorem of Phragmén-Lindelöf that

(∗)
$$\zeta(\sigma + it) = O\left(|t|^{\frac{1-\sigma}{2}+\varepsilon}\right) \quad \text{for any } \varepsilon > 0.$$

The *Lindelöf hypothesis* is that

$$\mu(\sigma) = \frac{1}{2} - \sigma \text{ if } \sigma \leq 1.$$

For $\sigma > \frac{1}{2}$ let $N(\sigma, T)$ be the number of zeroes ρ of $\zeta(s)$ with $\sigma \leq \operatorname{Re}\rho \leq 1$ and $0 \leq \operatorname{Im}\rho \leq T$, with multiplicities counted. We would like to prove the following relation between Lindelöf's hypothesis and the growth estimate of $N(\sigma, T)$. This equivalence statement says that if the Lindelöf hypothesis is true, then *essentially* all the nontrivial zeroes of the Riemann zeta function are on the critical line, because we know that the number of zeroes of the Riemann zeta function $\zeta(s)$ on $0 \leq \operatorname{Re}s \leq 1$ and $0 \leq \operatorname{Im}s \leq T$ is of the order $T\log T$.

Theorem (Equivalence Statement for the Lindelöf Hypothesis). Lindelöf's hypothesis holds if and only if

$$N(\sigma, T+1) - N(\sigma, T) = o\left(\log T\right)$$

for $\sigma > \frac{1}{2}$.

Proof. Assume that the Lindelöf hypothesis holds. Apply Jensen's theorem (stated and proved in an Appendix) to the holomorphic function $\zeta(s)$ on the disk
$$\left\{ s \in \mathbb{C} \,\middle|\, |s - (2 + iT)| < r \right\}$$

for $r < \frac{3}{2}$. Then

$$(\#) \qquad \sum_{|\rho - (2+iT)| < r} \log \frac{r}{|\rho - (2 + iT)|}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \zeta \left((2 + iT) + re^{i\theta} \right) \right| d\theta - \log |\zeta(2 + iT)|\,.$$

By (†) and (∗),

$$(\flat) \qquad \log \left| \zeta \left((2 + iT) + re^{i\theta} \right) \right| = o\,(\log T) \quad \text{for} \ \ 0 \le r < \frac{3}{2}.$$

Since, for any $0 < \delta < r$,

$$\sum_{|\rho - (2+iT)| < r - \delta} \log \frac{r}{|\rho - (2 + iT)|}$$

$$\ge \sum_{|\rho - (2+iT)| < r - \delta} \log \frac{r}{r - \delta}$$

$$= \left(\log \frac{r}{r - \delta} \right) \sum_{|\rho - (2+iT)| < r - \delta} 1 \,,$$

it follows from (#) and (♭) that

$$(\ddagger) \qquad \sum_{|\rho - (2+iT)| < r - \delta} 1 \ = o\,(\log T)$$

for $0 < \delta < r$. Since the set

$$\left\{ s = \sigma + it \,\bigg|\, \sigma_0 \le \sigma \le 1,\ T \le t \le T + 1 \right\}$$

with $\sigma_0 > \frac{1}{2}$ and $T_0 \in \mathbb{R}$ can be covered by a finite number of disks of the form

$$\left\{ s \in \mathbb{C}^n \,\Big|\, |s - (2 + iT_j)| < r_j - \delta_j \right\}$$

with $r_j > \frac{3}{2}$ and $\delta_j > 0$ and $T_j \in \mathbb{R}$, it follows from (‡) that

$$N(\sigma, T + 1) - N(\sigma, T) = o\,(\log T)$$

for $\sigma > \frac{1}{2}$.

We now prove the other direction of implication and assume that

$$N(\sigma, T+1) - N(\sigma, T) = o\left(\log T\right)$$

for $\sigma > \frac{1}{2}$. Choose $1 < r_1 < r_2 < r_3 < \frac{3}{2}$. We need the following slightly modified form of the approximate formula for the logarithmic derivative of the Riemann zeta Function. We now digress from our proof of the equivalence statement of the Lindelöf Hypothesis to take care of the modified form of the approximate formula for the logarithmic derivative of the Riemann zeta Function.

Approximate Formula for Logarithmic Derivative of Riemann Zeta Function.

(♮)
$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho - s_0| \leq r_3} \frac{1}{s - \rho} + O\left(\log T\right)$$

for $|s - s_0| \leq r_2$ if $s_0 = 2 + iT$.

Its proof will come from the following slightly modified form of the Landau lemma.

Landau's Lemma. Let $0 < r' < \frac{r}{2}$. Suppose $f(s)$ is a holomorphic function on $|s - s_0| \leq r$ and $M > 1$ such that

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M.$$

Let Z be the set of all zeroes of $f(s)$ on $|s - s_0| \leq \frac{r}{2}$ with multiplicities counted. Then

$$\left|\frac{f'(s)}{f(s)} - \sum_{\rho \in Z} \frac{1}{s - \rho}\right| < 8 \frac{(r + 2r')^2}{(r - 2r')^3} M$$

on $|s - s_0| \leq r'$.

Proof. Let

$$g(s) = \frac{f(s)}{\prod_{\rho \in Z}(s - \rho)}.$$

Then $g(s)$ is holomorphic on $|s - s_0| \leq r$ and is nowhere zero on $|s - s_0| < \frac{r}{2}$. On $|s - s_0| = r$ we have

$$|s - \rho| \geq \frac{r}{2} \geq |s_0 - \rho|$$

for $\rho \in Z$ and

$$\left| \frac{g(s)}{g(s_0)} \right| = \left| \frac{f(s)}{f(s_0)} \prod_{\rho \in Z} \left(\frac{s_0 - \rho}{s - \rho} \right) \right| \leq \left| \frac{f(s)}{f(s_0)} \right| < e^M,$$

which must hold also on $|s - s_0| \leq r$. Let

$$h(s) = \log \left(\frac{g(s)}{g(s_0)} \right)$$

on $|s - s_0| \leq \frac{r}{2}$ which is holomorphic and vanishes at s_0. Since $\operatorname{Re} h < M$ on $|s - s_0| \leq \frac{r}{2}$, it follows from the theorem of Borel-Caratheodory (with r and R replaced respectively by $\frac{\frac{r}{2} + r'}{2}$ and $\frac{r}{2}$) that

$$|h(s)| < \frac{\frac{r}{2} + r'}{\frac{r}{2} - \frac{\frac{r}{2} + r'}{2}} M = 2 \frac{r + 2r'}{r - 2r'} M$$

on $|s - s_0| \leq \frac{\frac{r}{2} + r'}{2}$. For $|s - s_0| \leq r'$,

$$|h'(s)| = \left| \frac{1}{2\pi i} \int_{|s - s_0| = \frac{\frac{r}{2} + r'}{2}} \frac{h(z)}{(z - s)^2} \, dz \right|$$

$$< \frac{\left(2 \frac{r + 2r'}{r - 2r'} \right) \left(\frac{\frac{r}{2} + r'}{2} \right)}{\left(\frac{r}{2} - \frac{\frac{r}{2} + r'}{2} \right)^2} M = 8 \frac{(r + 2r')^2}{(r - 2r')^3} M.$$

Q.E.D.

Proof of Approximate Formula for Logarithmic Derivative of Riemann Zeta Function. By the theorem on the growth rate of Riemann zeta function in terms of the imaginary part of its variable and by the theorem of Phragmén-Lindelöf, we have

$$|\zeta(\sigma + it)| = O\left(|t|^A\right)$$

for any fixed $\sigma \in \mathbb{R}$, where A is used as a generic symbol for a positive constant.

We apply Landau's lemma above to the case $f(s) = \zeta(s)$, $s_0 = 2 + iT$, $r = 2r_1$, $r' = r_2$, and $M = A \log T$. We use

$$\left| \frac{\zeta(s)}{\zeta(s_0)} \right| \leq T^A e^M = e^{A \log T}$$

for $|s - s_0| \leq 2r_1$ and get

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho - s_0| \leq r_3} \frac{1}{s - \rho} + O\left(\log T\right)$$

for $|s - s_0| \leq r_2$. Q.E.D.

Resumption of Proof of Equivalence Statement of the Lindelöf Hypothesis. Consider the holomorphic function

$$f(s) = \frac{\zeta'(s)}{\zeta(s)} - \sum_{|\rho - s_0| \leq r_3} \frac{1}{s - \rho}$$

on $|s - (2 + iT)| \leq r_2$. We are going to apply the three-circle theorem of Hadamard (stated and proved in an Appendix) to $f(s)$ and the three circles

$$\left\{ |s - (2 + iT)| = \frac{1}{2} \right\}, \quad \{ |s - (2 + iT)| = r_1 \}, \quad \{ |s - (2 + iT)| = r_2 \}.$$

For $|s - (2 + iT)| = \frac{1}{2}$ we have $\operatorname{Re} s \geq \frac{3}{2}$. Since $\operatorname{Re} \rho \leq 1$, it follows that

$$\left| \frac{1}{s - \rho} \right| \leq 2 \quad \text{for} \quad |s - (2 + iT)| = \frac{1}{2}.$$

By our assumption

$$N\left(2 - r_3, \, T + 2\right) - N\left(2 - r_3, \, T - 2\right) = o\left(\log T\right),$$

we have

$$\sum_{|\rho - s_0| \leq r_3} \frac{1}{s - \rho} = o\left(\log T\right) \quad \text{for} \quad |s - (2 + iT)| = \frac{1}{2}.$$

Since $\frac{\zeta'(s)}{\zeta(s)} = O\left(1\right)$ on $|s - (2 + iT)| = \frac{1}{2}$, it follows that

$$f(s) = o\left(\log T\right) \quad \text{for} \quad |s - (2 + iT)| = \frac{1}{2}.$$

We know from (♮) that

$$f(s) = O\left(\log T\right) \quad \text{for} \quad |s - (2 + iT)| = r_2.$$

By the three-circle theorem of Hadamard that

$$f(s) = (o\,(\log T))^{\lambda}\,(O\,(\log T))^{1-\lambda} = o\,(\log T) \quad \text{for} \quad |s - (2 + iT)| = r_1,$$

where
$$\lambda = \frac{\log r_2 - \log r_1}{\log r_2 - \log \frac{1}{2}}.$$

The maximum modulus principle gives

$$f(s) = o\,(\log T) \quad \text{for} \quad |s - (2 + iT)| \le r_1.$$

For $r_1 < \sigma < 1$ we integrate $f(s)ds$ from $\sigma + iT$ to $2 + iT$ to get

$$\log \zeta(\sigma + iT) - \log \zeta(2 + iT)$$
$$= \sum_{|\rho - (2+iT)| \le r_3} (\log((\sigma + iT) - \rho) - \log((2 + iT) - \rho)) + o\,(\log T).$$

From

$$\mathrm{Re}\,(\log((\sigma + iT) - \rho) - \log((2 + iT) - \rho))$$
$$= \log|(\sigma + iT) - \rho| - \log|(2 + iT) - \rho| < 0$$

for $|\rho - (2 + iT)| \le r_3$ and from $\zeta(2 + iT) = O\,(\log T)$ it follows that

$$\mathrm{Re}\,(\log \zeta(\sigma + iT)) < o\,(\log T).$$

This finishes the proof of the equivalence statement of the Lindelöf hypothesis.

Appendix. Jensen's Theorem

Theorem. Let $R > 0$ and $f(z)$ be a holomorphic function on $|z| < R$ with $f(0) \neq 0$. Let $r_1 \leq r_2 \leq r_3 \leq \cdots$ be the absolute values of all the zeroes of $f(z)$ on $|z| < R$. Then for $r_n \leq r \leq r_{n+1}$,

$$\log |f(0)| + \sum_{j=1}^{n} \log \frac{r}{r_j} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(re^{i\theta}\right) \right| d\theta.$$

Proof. First we consider the special case where $f(z)$ is nowhere zero on $|z| \leq r$. Then $\log |f(z)|$ is harmonic on $|z| \leq r$ and the mean-value property of harmonic functions gives

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(re^{i\theta}\right) \right| d\theta,$$

which is the statement of Jensen's theorem.

Secondly we consider the special case $f(z) = 1 - \frac{z}{ae^{i\alpha}}$ with $0 < a \leq r$ and $\alpha \in \mathbb{R}$. We have

$$\frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log \left| f\left(r\, e^{i\theta}\right) \right| d\theta = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log \left| 1 - \frac{r}{a} e^{i(\theta-\alpha)} \right| d\theta$$

$$= \log \frac{r}{a} + \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log \left| \frac{a}{r} e^{-i(\theta-\alpha)} - 1 \right| d\theta$$

$$= \log \frac{r}{a} + \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log \left| \frac{a}{r} e^{i(\theta-\alpha)} - 1 \right| d\theta = \log \frac{r}{a},$$

because

$$\int_{\theta=0}^{2\pi} \log \left| \frac{a}{r} e^{i(\theta-\alpha)} - 1 \right| d\theta$$

is the average of the harmonic function

$$\log \left| \frac{a}{r} z - 1 \right|$$

over the unit circle when $a < r$ and when $a = r$ we can argue by replacing a by $a - \delta$ for $\delta > 0$ and letting $\delta \to 0+$. Thus we have Jensen's theorem for the special case $f(z) = 1 - \frac{z}{a_1}$ with $a_1 \in \mathbb{C}$ and $|a_1| \leq r$.

Finally for the general case, we can write

$$f(z) = \left(1 - \frac{z}{a_1}\right) \cdots \left(1 - \frac{z}{a_n}\right) g(z)$$

with $g(z)$ nowhere zero on $|z| < r_{n+1}$ and $g(0) = f(0)$. The general case now follows from the preceding two special cases. Q.E.D.

Appendix. Hadamard's Three-Circle Theorem

Theorem. Let $0 < r_1 < r_2 < r_3$. Let $f(z)$ be a holomorphic function on $r_1 \leq |z| \leq r_3$. Let M_j be the supremum of $|F(z)|$ on $|z| = r_j$ for $j = 1, 2, 3$. Then

$$M_2^{\log \frac{r_3}{r_1}} \leq M_1^{\log \frac{r_3}{r_2}} M_3^{\log \frac{r_2}{r_1}}.$$

Proof. Let $\phi(z) = z^\lambda f(z)$, where λ is a constant to be determined later. Then $\phi(z)$ is a multi-valued holomorphic function on $r_1 \leq |z| \leq r_3$ with $|\phi(z)|$ single-valued and well-defined. By the maximum modulus principle,

$$|\phi(z)| \leq \max \left(r_1^\lambda M_1, r_3^\lambda M_3 \right).$$

Hence on $|z| = r_2$,

($\$$) $$|f(z)| \leq \max \left(r_1^\lambda r_2^{-\lambda} M_1, r_3^\lambda r_2^{-\lambda} M_3 \right).$$

Choose

$$\lambda = -\frac{\log \frac{M_3}{M_1}}{\log \frac{r_3}{r_1}}$$

so that

$$r_1^\lambda M_1 = r_3^\lambda M_3.$$

For this choice of λ it follows from ($\$$) that

$$M_2 \leq \left(\frac{r_2}{r_1} \right)^{-\lambda} M_1$$

and

$$M_2^{\log \frac{r_3}{r_1}} \leq \left(\frac{r_2}{r_1} \right)^{\log \frac{M_3}{M_1}} M_1^{\log \frac{r_3}{r_1}} \leq M_1^{\log \frac{r_3}{r_2}} M_3^{\log \frac{r_2}{r_1}}.$$

Q.E.D.