Mean-Value and Oscillation Theorems

The mean-value theorems replace the limit by an equal expression before the taking of limit. For derivatives which are the limits of difference quotients, the mean-value theorem in derivative form says that

$$\phi'(c) = \frac{\phi(b) - \phi(a)}{b - a}$$

for some $a < c < b$, which equates the limit $\phi'(c)$ at some point c to the difference quotient

$$\frac{\phi(b) - \phi(a)}{b - a}$$

before the taking of limit. In integral form

$$\int_a^b \phi(x)dx = \phi(c)(b - a)$$

for some c between a and b, which equals the area under a graph to a rectangle whose base is the interval of integration $[a, b]$ and whose height is the functional value $\phi(c)$ at some point c of the integral $[a, b]$.

When $\phi(x)$ is poistive and non-increasing, in equating the area under the graph to a rectangle, we can specify the height to be $\phi(a)$ (which is the functional value of ϕ at the starting end-point a) instead of some undetermined $\phi(c)$ but the base of the rectangle would be $c - a$ for some c between a and b. We have

$$\int_a^b \phi(x)dx = \phi(a)(c - a)$$

for some c between a and b. The following second mean-value theorem describes such a situation when the setting is the more general Riemann-Stieltjes integral instead of the Riemann integral. The measure used for an interval $[\alpha, \beta]$ is $\int_\alpha^\beta f(x)dx$ instead of just $\beta - \alpha$.

Second Mean Value Theorem. If ϕ is positive and non-increasing, then

$$\int_a^b f(x)\phi(x)dx = \phi(a)\int_a^\xi f(x)dx$$

for some ξ between a and b.

Proof. Let $F'(x) = f(x)$ with $F(a) = 0$ and let m (respectively M) be the infimum (respectively supremum) of F over $[a, b]$. Then

$$\int_{x=a}^{b} f(x)\phi(x)dx = F(x)\phi(x)\Big|_{x=a}^{x=b} - \int_{x=a}^{b} F(x)\phi'(x)dx$$

$$= F(b)\phi(b) + \int_{x=a}^{b} F(x)\left(-\phi'(x)\right)dx$$

$$\begin{cases} \leq F(b)\phi(b) + \int_{x=a}^{b} M\left(-\phi'(x)\right)dx \\ \geq F(b)\phi(b) + \int_{x=a}^{b} m\left(-\phi'(x)\right)dx \end{cases}$$

$$= \begin{cases} F(b)\phi(b) - M\phi(b) + M\phi(a) \\ F(b)\phi(b) - m\phi(b) + m\phi(a) \end{cases}$$

$$= \begin{cases} \left(F(b) - M\right)\phi(b) + M\phi(a) & \leq M\phi(a) \\ \left(F(b) - m\right)\phi(b) + m\phi(a) & \geq m\phi(a) \end{cases}$$

By the intermediate value theorem there exists some ξ between a and b such that

$$\int_{x=a}^{b} f(x)\phi(x)dx = \phi(a)\int_{x=a}^{\xi} f(x)dx.$$

Q.E.D.

Symmetric Form of the Second Mean Value Theorem. If ϕ is non-increasing, then

$$\int_{x=a}^{b} f(x)\phi(x)dx = \phi(a)\int_{x=a}^{\xi} f(x)dx + \phi(b)\int_{x=\xi}^{b} f(x)dx.$$

Proof. Applying the above Second Mean-Value Theorem to $\phi(x) - \phi(b)$ instead of to $\phi(x)$, we get

$$\int_{x=a}^{b} f(x)\left(\phi(x) - \phi(b)\right)dx = \left(\phi(a) - \phi(b)\right)\int_{x=a}^{\xi} f(x)dx.$$

Q.E.D.

We now introduce some oscillation theorems which are consequences of the above integral mean-value theorems. The main point of these oscillation theorems is that, with a suitable growth rate for $F(x)$ and a suitable uniform bound for $G(x)$, there is a bound for the integral $\int_a^b e^{iF(x)}dx$ or $\int_a^b G(x)e^{iF(x)}dx$ which is independent of a and b. The reason is the cancelation coming from $e^{iF(x)}$ with different values of x.

Simpler Form of Oscillation Lemma. Let F be real-valued with F' monotone such that $F'(x) \geq m > 0$ (or $F'(x) \leq -m < 0$) on $[a, b]$. Then

$$\left| \int_{x=a}^{b} e^{iF(x)}dx \right| \leq \frac{4}{m}.$$

Proof. We do the case $\frac{1}{F'}$ non-increasing and $F'(x) \geq m > 0$. The other cases are analogous.

$$\begin{aligned} \int_{x=a}^{b} \cos\left(F(x)\right) dx &= \int_{x=a}^{b} \frac{F'(x)\cos F(x)}{F'(x)} dx \\ &= \frac{1}{F'(a)} \int_{x=a}^{\xi} F'(x)\cos F(x) dx \\ &= \frac{\sin\left(F(\xi)\right) - \sin\left(F(a)\right)}{F'(a)} \end{aligned}$$

which is no more than $\frac{2}{m}$ in absolute value. A similar estimate holds when cosine is replaced by sine. Q.E.D.

More General Oscillation Lemma. Let F and G be real-valued functions with $\frac{F'}{G}$ monotone such that $\frac{F'}{G} \geq m > 0$ (or $\frac{F'}{G} \leq -m < 0$) on $[a, b]$. Then

$$\left| \int_{x=a}^{b} G(x)e^{iF(x)}dx \right| \leq \frac{4}{m}.$$

Proof. We do the case $\frac{G}{F'}$ non-increasing and $\frac{F'}{G} \geq m > 0$. The other cases are analogous.

$$\begin{aligned} \int_{x=a}^{b} G(x)\cos\left(F(x)\right) dx &= \int_{x=a}^{b} \frac{G}{F'}\, F'(x)\cos F(x)\, dx \\ &= \frac{G(a)}{F'(a)} \int_{x=a}^{\xi} F'(x)\cos F(x) dx \end{aligned}$$

$$= \frac{G(a)}{F'(a)} \left(\sin \left(F(\xi) \right) - \sin \left(F(a) \right) \right)$$

which is no more than $\frac{2}{m}$ in absolute value. A similar estimate holds when cosine is replaced by sine. Q.E.D.

Oscillation Lemma Involving Second Derivative. Let F be a real-valued twice continuously differentiable function with $F''(x) \geq r > 0$ on $[a, b]$. Let $G(x)$ be a real-valued function such that $\frac{G(x)}{F'(x)}$ is monotone and $|G(x)| \leq M$ on $[a, b]$. Then

$$\left| \int_{x=a}^{b} G(x) e^{iF(x)} dx \right| \leq \frac{8M}{\sqrt{r}}.$$

(The idea of the lemma is that the inquality $F''(x) \geq r > 0$ guarantees enough oscillation to make the estimate independent of a and b.)

Proof. From $F''(x) \geq r > 0$ it follow that $F'(x)$ is strictly increasing and there is at most one root in the interval (a, b). We assume that there is in fact one root c for $F'(x)$ in the interval (a, b). The case of no root is actually simpler and can easily be adapted from the case of one root. Choose some δ which will be specified later. We assume that $a + \delta \leq c \leq b - \delta$. Write

$$\int_{x=a}^{b} G(x) e^{iF(x)} dx = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{x=a}^{c-\delta} G(x) e^{iF(x)} dx,$$

$$I_2 = \int_{x=c-\delta}^{c+\delta} G(x) e^{iF(x)} dx,$$

$$I_3 = \int_{x=c+\delta}^{b} G(x) e^{iF(x)} dx.$$

For $c \leq x \leq b$,

$$F^prime(x) = \int_{c}^{x} F''(t) dt \geq r(x - c) \geq r\delta.$$

From $|G(x)| \leq M$ it follows that

$$\frac{F'(x)}{G(x)} \geq \frac{r\delta}{M} \quad \text{or} \quad \frac{F'(x)}{G(x)} \leq -\frac{r\delta}{M} \quad \text{for } c \leq x \leq b.$$

By the More General Oscillation Lemma given above,

$$|I_3| \leq \frac{4M}{r\delta}.$$

Likewise,

$$|I_1| \leq \frac{4M}{r\delta}.$$

From $|G(x)| \leq M$ it follows that

$$|I_2| \leq 2M\delta.$$

Thus

$$\left|\int_{x=a}^{b} G(x) e^{iF(x)} dx\right| \leq \frac{4M}{r\delta} + 2M\delta.$$

We now set $\delta = \frac{2}{\sqrt{r}}$ and get

$$\left|\int_{x=a}^{b} G(x) e^{iF(x)} dx\right| \leq \frac{8M}{\sqrt{r}}.$$

The case $c < a + \delta$ or $c > b - \delta$ are similar. Q.E.D.