Overview of the Material on the Riemann Hypothesis to Be Covered in the Course

The goal of this topic is to introduce some of major function-theoretical techniques which were historically developed in attempts to solve the Riemann hypothesis. So far as this topic is concerned, we will cover the following.

(1) Two proofs of the *functional equation* of the Riemann zeta function:

   (i) Using the representation of the Riemann zeta function as a contour integral and expressing it as a sum of residues as the contour expands to infinity.

   (ii) Using Poisson summation formula, the Gauss error function, and Mellin transform.

(2) *Dirichlet series.* The abscissa of convergence (or absolute convergence) for a Dirichlet series is the analog of the radius convergence for a power series. One important difference is the gap between the abscissa of convergence and the abscissa of absolute convergence. The gap can achieve the maximum value of 1. The interesting case for us is

$$\left(1 - \frac{1}{2^{s-1}}\right)\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

whose abscissa of convergence is $\operatorname{Re} s = 0$ and whose abscissa of absolute convergence is $\operatorname{Re} s = 1$.

(3) The *Perron formula* expresses the coefficient of a Dirichlet series by an integral along a vertical line. It is the analog of the Cauchy formula for the coefficients of a power series expressed as an integral over a circle. It is the also the analog of the formula which expresses the coefficient of a Fourier series in terms of an integral over an interval.

(4) Application of the Perron formula to the Dirichlet series

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{n^s},$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^e,\, e \in \mathbb{N},\, e \geq 1,\, p \text{ prime} \\ 0 & \text{otherwise.} \end{cases} .$$

The application, together with the summation of the residues from the integral over an expanding rectangle, gives us an *explicit formula* (or *approximate formula*) for

$$\sum_{n<x} \Lambda(n)$$

in terms of x.

(4) Relation between the set Z of nontrivial zeroes of $\zeta(s)$ (*i.e.,* those in $0 < \operatorname{Re} s < 1$) and the error of approximating $\sum_{n<x} \Lambda(n)$ by x. We will prove the following two statements.

(i) $Z \subset \{\operatorname{Re} s \leq \theta\}$ for some $0 < \theta < 1$ implies

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^\theta (\log x)^2\right).$$

(ii)

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^\alpha\right).$$

for some $0 < \alpha < 1$ implies that $Z \subset \{\operatorname{Re} s \leq \alpha\}$.

The Riemann hypothesis is equivalent to the error estimate

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^{\frac{1}{2}} (\log x)^2\right).$$

(5) Hadamard's proof the *prime number theorem* which will derived from

$$Z \subset \left\{\operatorname{Re} s < 1 - \gamma \min\left(1,\, \frac{1}{\log \operatorname{Im} s}\right)\right\}$$

for some $\gamma > 0$. This relation is weaker than $Z \subset \{\operatorname{Re} s \leq \alpha\}$ for some $0 < \alpha < 1$ which is listed above in the relation between Z and the error in approximating $\sum_{n<x} \Lambda(n)$ by x. Hadamard's factorization theorem for entire functions of finite order is an important ingredient in the proof.

(6) The theorem of Hardy-Littlewood on infinite number of zeros of $\zeta(s)$ on $\operatorname{Re} s = \frac{1}{2}$. The proof is based on comparing the growth of

$$\int_{t=T}^{2T} |Z(t)|\, dt \quad \text{and} \quad \int_{t=T}^{2T} Z(t) dt$$

as $T \to \infty$, where the real-valued function $Z(t)$ is defined by

$$Z(t) = -2\pi^{\frac{1}{4}} \frac{\Xi(t)}{\left(t^2 + \frac{1}{4}\right) \left|\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right)\right|}$$

with

$$\Xi(t) = \xi\left(\frac{1}{2} + it\right)$$

and

$$\xi(s) = \frac{s(1-s)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

The tools involve the growth behavior of $\zeta(s)$ along vertical lines and the Phragmén-Lindelöf theorem which can be regarded as an analog of the Hadamard three-circle theorem in the case of vertical strips.

(6) If time permits, we will discuss the theorem of Selberg that there exists some $c > 0$ so that inside $\{T \leq \operatorname{Im} s \leq 2T\}$ for T sufficiently large the number of zeros of $\zeta(s)$ on $\operatorname{Re} s = \frac{1}{2}$ is at least c times the total number inside $\{T \leq \operatorname{Im} s \leq 2T\}$.