Prime Number Theorem

The zeta function is given by the Euler product formula

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}.$$

Taking logarithmic derivative yields

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p \text{ prime}} \frac{-p^{-s} \log \frac{1}{p}}{1 - \frac{1}{p^s}} = \sum_{p \text{ prime}} \frac{\frac{\log p}{p^s}}{1 - \frac{1}{p^s}}$$

$$= \sum_{p \text{ prime}} (\log p) \left(\frac{1}{p^s} \sum_{k=0}^{\infty} \frac{1}{p^{ks}} \right) = \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{n^s},$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \ k \in \mathbb{N}, e \geq 1, p \text{ prime} \\ \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$3 + 4 \cos t + \cos 2t = 2 \left(1 + \cos^2 t \right)^2 \geq 0,$$

it follows that for $\sigma > 1$,

$$\text{Re} \left(-3 \frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4 \frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} - \frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)} \right)$$

$$= \sum_{p \text{ prime}} \sum_{k=1}^{\infty} \text{Re} \left(3 p^{-k\sigma} + 4 p^{-k\sigma - ikt} + p^{-k\sigma - i2kt} \right) \log p$$

$$= \sum_{p \text{ prime}} \sum_{k=1}^{\infty} p^{-k\sigma} \left(3 + 4 \cos t + \cos tx \right) \log p$$

$$= \sum_{p \text{ prime}} \sum_{k=1}^{\infty} p^{-k\sigma} 2 \left(1 + \cos^2 t \right)^2 \log p \geq 0.$$

Recall the following estimate for the logarithmic derivative of Riemann zeta function.

Approximate Formula for Logarithmic Derivative of Riemann Zeta Function.
If $\rho = \beta + i\gamma$ runs through the zeroes of $\zeta(s)$, then

$$(*) \qquad \frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} + O\left(\log|t|\right),$$

uniformly for $-1 \leq \sigma \leq 2$, where $s = \sigma + it$. Moreover, for any given $C > 0$ the number of roots ρ of $\zeta(s)$ with $0 \leq \mathrm{Re}\,\rho \leq 1$ and $|\mathrm{Im}\,\rho - T| \leq C$ is no more than $O\left(\log T\right)$.

For $\sigma > 1$ and $0 \leq \mathrm{Re}\,\rho \leq 1$, we have

$$(\dagger) \qquad \mathrm{Re}\,\frac{1}{s-\rho} = \frac{\sigma - \mathrm{Re}\,\rho}{|s-\rho|^2} > 0.$$

We now fix an arbitrary $\rho_1 \in Z$ and let $t = \mathrm{Im}\,\rho_1$. By taking the real part of $(*)$ and using $(\dagger)$, we obtain

$$\mathrm{Re}\left(-\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right) = -\mathrm{Re}\sum_{|t-\gamma|\leq 1}\frac{1}{s-\rho} + O\left(\log t\right)$$

$$< -\frac{1}{\sigma-\rho_1} + O\left(\log|t|\right) \quad \text{for} \quad \sigma > 1.$$

Using $(*)$ and $(\dagger)$, we also get

$$\mathrm{Re}\left(-\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right) = -\mathrm{Re}\sum_{|t-\gamma|\leq 1}\frac{1}{s-\rho} + O\left(\log|t|\right) < O\left(\log|t|\right).$$

Finally from the simple pole of $\zeta(s)$ at $s = 1$ we get

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma-1} + \text{constant} \quad \text{for} \quad \sigma > 1.$$

Adding up the three estimates, we get

$$\mathrm{Re}\left(-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} - \frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right)$$

$$< -\frac{3}{\sigma-1} + \frac{4}{\sigma-\mathrm{Re}\,\rho_1} + O\left(\log|t|\right).$$

Setting $\sigma = 1 + (1 - \operatorname{Re} \rho_1)$, we obtain

$$-\frac{3}{\sigma - 1} + \frac{4}{\sigma - \operatorname{Re} \rho_1} = \frac{-1}{1 - \operatorname{Re} \rho_1}$$

and consequently

$$\operatorname{Re}\left(-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} - \frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)}\right) < \frac{-1}{1 - \operatorname{Re} \rho_1} + O(\log |t|)$$

for $\sigma = 1 + (1 - \operatorname{Re} \rho_1)$. From

$$\operatorname{Re}\left(-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} - \frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)}\right) \geq 0$$

it follows that

$$\frac{-1}{1 - \operatorname{Re} \rho_1} + O(\log |t|) \geq 0$$

which means that there exists some $C > 0$ such that

$$\frac{1}{1 - \operatorname{Re} \rho_1} \leq C \log |t|$$

or

$$\operatorname{Re} \rho_1 < 1 - \frac{1}{C \log |\operatorname{Im} \rho_1|}.$$

Since ρ_1 is just an arbitrary element of Z, we conclude that there exists some positive number C such that

$$\operatorname{Re} \rho < 1 - \frac{1}{C \log |\operatorname{Im} \rho|}$$

for every $\rho \in Z$.

We now use the explicit formula for the logarithmic derivative of the Riemann zeta function.

$$\sum_{n < x} \Lambda(n) = x - \sum_{\substack{\rho \in Z, \\ |\operatorname{Im} \rho| < T}} \frac{x^\rho}{\rho} + \text{error term},$$

where the error term is

$$O\left(\frac{x (\log x)^2}{T}\right) + O\left(\frac{x \log x}{\langle x \rangle T}\right) + O\left(\frac{x \log T}{T}\right),$$

where $\langle x \rangle$ is the distance between x and the closest integer. The error term becomes

$$O\left(x\, e^{-\hat{c}\sqrt{\log x}}\right)$$

for some positive $\hat{c} > 0$, when we choose T by $\log T = \sqrt{\log x}$ and $\langle x \rangle = \frac{1}{2}$. What is important is the estimate for the term

$$\sum_{\substack{\rho \in Z, \\ |\operatorname{Im}\rho| < T}} \frac{x^\rho}{\rho}.$$

Now

$$\left|x^\rho\right| \leq x^{1-\frac{1}{C\log|\operatorname{Im}\rho|}} \leq x^{1-\frac{1}{C\log T}} \leq x^{1-\frac{1}{C\sqrt{\log x}}} = x e^{-\frac{\sqrt{\log x}}{C}}.$$

To estimate

$$\sum_{\substack{\rho \in Z, \\ |\operatorname{Im}\rho| < T}} \frac{1}{\rho},$$

we divide up the interval $[-T, T]$ into subintervals of length 1. Inside each interval $[m, m+1]$ there at most $O\left(\log|m|\right)$ elements of Z. Thus

$$\sum_{\substack{\rho \in Z, \\ |\operatorname{Im}\rho| < T}} \frac{1}{\rho} \leq O\left(\sum_{m=1}^{T} \frac{\log m}{m}\right) = O\left((\log T)^2\right) = O\left(\log x\right)$$

from comparing sums over integers to approximating integrals. So we get the estimate

$$\sum_{n<x} \Lambda(n) = x + O\left(x e^{-c\sqrt{\log x}}\log x\right),$$

which implies

$$(\#) \qquad \sum_{n<x} \Lambda(n) = x + O\left(x e^{-c'\sqrt{\log x}}\right)$$

for some $c' > 0$, because the order of $e^{c\sqrt{\log x}}$ is higher than any positive power of $\sqrt{\log x}$.

From Error Estimate to Prime Number Distribution. We break up the summation $\sum_{n<x} \Lambda(n)$ into a part involving just prime numbers and another part involving higher powers of prime numbers as follows.

$$\sum_{n<x} \Lambda(n) = \sum_{\substack{p \text{ prime}, \\ p^k < x}} \log p = \sum_{\substack{p \text{ prime}, \\ p < x}} \log p + \sum_{\substack{p \text{ prime}, \ p^k < x \\ \text{for some } k \geq 2}} \log p.$$

We can estimate the second sum as follows.

$$\sum_{\substack{p \text{ prime, } p^k < x \\ \text{for some } k \geq 2}} \log p \leq \sum_{2 \leq p \leq \sqrt{x}} \log p \leq \sqrt{x} \log x$$

$$\leq x \, e^{-\frac{1}{2} \log x + \log \log x} = O\left(x \, e^{-\tilde{C}\sqrt{\log x}}\right)$$

for any positive number $\tilde{C}$. From $(\#)$ it follows that

$$\sum_{p \text{ prime, } p < x} \log p = x + O\left(xe^{-\hat{c}\sqrt{\log x}}\right) \quad \text{for some } \hat{c} > 0.$$

Let

$$A(x) = \sum_{p \text{ prime, } p \leq x} \log p$$

so that

$$(\#\#) \qquad\qquad A(x) = x + O\left(xe^{-\hat{c}\sqrt{\log x}}\right).$$

Recall the notation

$$\pi(x) = \sum_{p \text{ prime, } p \leq x} 1$$

which is the number of primes not exceeding x. Using summation by parts and $(\#\#)$, we get

$$\pi(x) = \sum_{p \text{ prime, } p \leq x} 1 = \sum_{n \leq x} \frac{A(n+1) - A(n)}{\log n} = \sum_{3 \leq n \leq x} A(n) \left(\frac{1}{\log(n-1)} - \frac{1}{\log n}\right)$$

$$= \sum_{3 \leq n \leq x} \left(n + O\left(ne^{-\hat{c}\sqrt{\log n}}\right)\right) \left(\frac{1}{\log(n-1)} - \frac{1}{\log n}\right)$$

$$= \sum_{3 \leq n \leq x} n \left(\frac{1}{\log(n-1)} - \frac{1}{\log n}\right) + O\left(xe^{-\hat{c}\sqrt{\log x}} \sum_{3 \leq n \leq x} \left(\frac{1}{\log(n-1)} - \frac{1}{\log n}\right)\right)$$

$$= \sum_{3 \leq n \leq x} n \left(\frac{1}{\log(n-1)} - \frac{1}{\log n}\right) + O\left(xe^{-\hat{c}\sqrt{\log x}}\right)$$

$$= \sum_{3 \leq n \leq x} \frac{1}{\log n} + O\left(xe^{-\hat{c}\sqrt{\log x}}\right)$$

$$= \int_2^x \frac{du}{\log u} + O\left(x e^{-\hat{c}\sqrt{\log x}}\right)$$

$$= \frac{x}{\log x} + \int_2^x \frac{du}{(\log u)^2} + O\left(x e^{-\hat{c}\sqrt{\log x}}\right).$$

Finally from

$$\int_2^x \frac{du}{(\log u)^2} = \int_2^{\sqrt{x}} \frac{du}{(\log u)^2} + \int_{\sqrt{x}}^x \frac{du}{(\log u)^2}$$

$$\leq \frac{\sqrt{x} - 2}{(\log 2)^2} + \frac{x - \sqrt{x}}{(\log \sqrt{x})^2} = O\left(\frac{x}{(\log x)^2}\right)$$

we conclude that

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

which is the Prime Number Theorem.