# THE THEOREM OF RIEMANN-ROCH AND ABEL's THEOREM

§1. *Introduction*

On $\mathbb{C}$ the dimension of the vector space of polynomials of degree $d$ is $d+1$. When we consider polynomials as meromorphic functions on the Riemann sphere it means that the dimension of the vector space of meromorphic functions whose poles are at most poles of order $\leq d$ at infinity is $d+1$. In general when we specify $k$ points $A_1, \cdots, A_k$ on the Riemann sphere the vector space of meromorphic functions whose poles are at most of order $d_\nu$ at $A_\nu$ has dimension

$$1 + \sum_{\nu=1}^{k} d_\nu,$$

because any such function is of the form

$$c + \sum_{\nu=1}^{k} \sum_{\mu=1}^{d_\mu} \frac{c_{\mu\nu}}{(z - A_\nu)^\mu}$$

for some constants $c$, $c_{\mu\nu}$ when each $A_\nu$ is a finite point and when $A_\nu$ is the infinity point

$$\sum_{\mu=1}^{d_\mu} \frac{c_{\mu\nu}}{(z - A_\nu)^\mu}$$

is to be replaced by

$$\sum_{\mu=1}^{d_\mu} c_{\mu\nu} z^\mu.$$

When one has a compact Riemann surface of genus $g$ instead of the Riemann sphere can one say anything about the dimension of the vector space of all meromorphic functions on the Riemann sphere? As an example let us look at the case of complex tori. On the complex torus the elliptic functions are meromorphic functions. In this case when we allow only at most a simple pole at a single given point we have only the constant functions as meromorphic functions because the sum of the residues of a meromorphic function on a torus must vanish. When we allow at most double pole at a single given point the vector space of meromorphic functions has dimension 2, because it is spanned by a nonzero constant function and the Weierstrass $\mathcal{P}$ function.

The theorem of Riemann-Roch is an answer to this question for the general case.

§2. *Divisors and the statement of the theorem of Riemann-Roch*

To specify the number of points allowed as poles and the maximum order allowed for each pole we introduce the notion of a divisor. A *divisor* of a Riemann surface $M$ is a finite formal linear combination of points of $M$ with integer coefficients. In other words consider the abelian group generated by points of $M$ and the divisors of $M$ are simply elements of this abelian group. So a divisor is of the form

$$\sum_{\nu=1}^{k} m_\nu P_\nu,$$

where $m_\nu \in \mathbb{Z}$ and $P_\nu \in M$. The *degree* of the divisor $\sum_{\nu=1}^{k} m_\nu P_\nu$ is defined as $\sum_{\nu=1}^{k} m_\nu$. We say that the divisor $\sum_{\nu=1}^{k} m_\nu P_\nu$ is nonnegative if each $m_\nu$ is nonnegative. We can perform additions and subtractions on divisors by performing the operations on the coefficients. For two divisors $D_1$ and $D_2$ we say that $D_1 \geq D_2$ if $D_1 - D_2$ is a nonnegative divisor.

When we are given a meromorphic function $f$ there is a natural divisor associated to $f$ which we denote by $\operatorname{div} f$. This divisor $\operatorname{div} f$ is given as follows. Suppose $P_1, \cdots, P_k$ are all the zeroes of $f$ with orders $m_1, \cdots, m_k$ and $Q_1, \cdots, Q_\ell$ are all the zeroes of $f$ with orders $n_1, \cdots, n_\ell$. Then

$$\operatorname{div} f = \sum_{\nu=1}^{k} m_\nu P_\nu - \sum_{\nu=1}^{\ell} n_\nu Q_\nu.$$

In the case of a compact Riemann surface the degree of $\operatorname{div} f$ is always zero, because $d \log f$ is a meromorphic form and the sum of the residues of any meromorphic form on a compact Riemann surface is zero.

We can also define the divisor $\operatorname{div} \alpha$ of a meromorphic form on a Riemann surface in the same way. Suppose $P_1, \cdots, P_k$ are all the zeroes of $\alpha$ with orders $m_1, \cdots, m_k$ and $Q_1, \cdots, Q_\ell$ are all the zeroes of $\alpha$ with orders $n_1, \cdots, n_\ell$. Then

$$\operatorname{div} \alpha = \sum_{\nu=1}^{k} m_\nu P_\nu - \sum_{\nu=1}^{\ell} n_\nu Q_\nu.$$

Suppose $f$ is a meromorphic function on a Riemann surface $M$ and $\alpha$ is a meromorphic form on $M$ and $D = \sum_{\nu=1}^{k} m_\nu P_\nu$ is a divisor of $M$ with each

$m_\nu > 0$. Then the statement $\operatorname{div} f \geq -D$ simply says that the poles of $f$ in the Riemann surface $M$ are at most of order $m_\nu$ at $P_\nu$ $(1 \leq \nu \leq k)$. The statement $\operatorname{div} \alpha \geq -D$ simply says that all the poles of $\alpha$ in the Riemann surface $M$ are at most of order $m_\nu$ at $P_\nu$ $(1 \leq \nu \leq k)$. The statement $\operatorname{div} \alpha \geq D$ simply says that $\alpha$ is holomorphic and vanishes to order at least $m_\nu$ at $P_\nu$ $(1 \leq \nu \leq k)$.

Now we are ready to state the theorem of Riemann-Roch. Let $M$ be a compact Riemann surface of genus $g$ and $D$ be a divisor of $M$ of degree $m$. Let $\mathcal{A}$ be the vector space of all mermorphic functions $f$ on $M$ with $\operatorname{div} f \geq -D$ and let $\mathcal{B}$ be the vector space of all meromorphic forms $\alpha$ on $M$ with $\operatorname{div} \alpha \geq D$. Let $a$ be the complex dimension of $\mathcal{A}$ and $b$ be the complex dimension of $\mathcal{B}$. The theorem of Riemann-Roch states that $a-b = m-g+1$. In particular since $b$ is nonnegative we have $a \geq m - g + 1$ and it gives us a lower bound on the dimension of meromorphic functions with poles allowed only at specified points to no more than specified orders.

§3. *The proof of the theorem of Riemann-Roch for nonnegative divisor.*

We first prove the theorem of Riemann-Roch for the case of nonnegative divisors. Earlier we constructed meromorphic forms $dE_\ell(P)$, $dF_\ell(P)$ on compact Riemann surfaces by using Dirichlet integrals. The idea of the proof of the theorem of Riemann-Roch is that we can use these forms to form a basis of meromorphic forms *without simple poles* and we compute the dimension of the vector space $\mathcal{A}$ of meromorphic functions by taking the differentials of these meromorphic functions and using a basis constructed from the meromorphic forms $dE_\ell(P)$, $dF_\ell(P)$. Let us change our definitions of $dE_\ell(P)$, $dF_\ell(P)$ to make the periods real instead of purely imaginary. We denote by $dE_\ell(P)$ (respectively $dF_\ell(P)$) the meromorphic form whose principal part at $P$ is given by $\frac{dz}{z^{\ell+1}}$ (respectively $\frac{i\,dz}{z^{\ell+1}}$) in some holomorphic local coordinate system $z$ centered at $P$ whose periods are all real.

Let the divisor $D$ be $\sum_{k=1}^n \ell_k P_k$ with $\ell_k$ positive. For arbitrary real numbers $\lambda_{k\ell}$, $\mu_{k\ell}$ $(1 \leq \ell \leq \ell_k,\ 1 \leq k \leq n)$ consider the meromorphic form

$$(*) \qquad d\varphi = \sum_{k=1}^{n} \sum_{\ell=1}^{\ell_k} \left(\lambda_{k\ell} dE_\ell(P_k) + \mu_{k\ell} dF_\ell(P_k)\right).$$

The differential $df$ of any meromorphic function $f$ with $\operatorname{div} f \geq -D$ must be of the form $d\varphi$ as one can easily see by comparing the principal parts and arguing that the holomorphic form $d\varphi - df$ with only real periods must be identically zero. On the other hand, if the periods of $d\varphi$ are all zero then we can integrate $d\varphi$ of the form $(*)$ to obtain a meromorphic function $f$ with $\operatorname{div} f \geq -D$. So to determine the dimension a of $\mathcal{A}$ it is the same as determining the dimension of the set of all $d\varphi$ with zero periods and then add 1 to the result (to take into account all the constant functions).

*Relation Between Vanishing Periods and Contour Integrals.* We use the following method to detect the vanishing of the periods of $d\varphi$. Since the theorem of Riemann-Roch for the case of the Riemann sphere can very easily be verified directly we assume from now on that the genus $g$ is at least 1. Take a holomorphic form $dw$ on the Riemann sphere so that $w$ is a single-valued holomorphic function on the fundamental polygon. Let $C$ be the boundary of the fundamental polygon. Then

$$\frac{1}{2\pi i}\int_C \varphi dw = \frac{1}{2\pi i}\,[\varphi]' J[w],$$

where $J$ is the $2g \times 2g$ skew-symmetric matrix

$$\begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}$$

and $[\varphi]$ (respectively $[w]$ is the column $(2g)$-vector whose entries are the periods

$$\int_{A_k} \varphi \quad (1 \leq k \leq g), \qquad \int_{B_k} \varphi \quad (1 \leq k \leq g)$$

and, likewise, $[dw]$ (respectively $[w]$ is the column $(2g)$-vector whose entries are the periods

$$\int_{A_k} dw \quad (1 \leq k \leq g), \qquad \int_{B_k} dw \quad (1 \leq k \leq g).$$

This is obtained by integrating along the boundary of the $(4g)$-gon and group together the integration over the four sides from two conjugate loops.

More precisely, take two meromorphic forms $d\xi$ and $d\eta$ on the Riemann surface $M$, where $\xi$ and $\eta$ are two meromorphic functions on the fundamental $4g$-gon. (In our case, $\xi$ is $\varphi$ and $\eta$ is $dw$.) We have conjugate crosscuts $A_k$,

$B_k$ $(1 \leq k \leq g)$ so that the sides of the fundamental polygon in the counter-clockwise sense are

$$A_1, B_1, -A_1, -B_1, \cdots, A_g, B_g, -A_g, -B_g.$$

We denote the period of $d\xi$ along $A_k$ by $\xi(A_k)$ which is equal to the difference of the value of $\xi$ at the two end-points of $A_k$. The notations $\xi(B_k)$, $\eta(A_k)$, $\eta(B_k)$ have similar meanings. Let $\Omega$ denote the fundamental polygon. By Cauchy's theorem

$$\int_{\partial\Omega} \xi d\eta = \sum_{k=1}^{g} \left( \int_{A_k} \{\xi - (\xi + \xi(B_k))\} d\eta + \int_{B_k} \{\xi - (\xi - \xi(A_k))\} d\eta \right)$$

$$= \sum_{k=1}^{g} \left( \xi(A_k) \int_{B_k} d\eta - \xi(B_k) \int_{A_k} d\eta \right) = \sum_{k=1}^{g} \left( \xi(A_k)\eta(B_k) - \xi(B_k)\eta(A_k) \right).$$

The reason why we have the integrand $\{\xi - (\xi + \xi(B_k))\} d\eta$ along $A_k$ is the following. This integral comes from the sum of the integral over $A_k$ and over $-A_k$ and from grouping together the value $\xi(z)$ at a point $z$ of $A_k$ and the value $\xi(z')$ at the corresponding $z'$ of $-A_k$. We have $\xi(z') = \xi(z) + \xi(B_k)$, because we can compare $\xi(z')$ with $\xi(z)$ by going along $\partial\Omega$ in the counter-clockwise sense from the point $x$ of $A_k$ along $A_k$ to the juncture of $A_k$ and $B_k$ and then along $B_k$ to the juncture of $B_k$ and $-A_k$ and then along $-A_k$ to $x'$, with the contribution from along $A_k$ cancelling the contribution from along $-A_k$ precisely. Since $A_k$ is in the positive sense and $-A_k$ has the opposite sense, so we end up with $\xi(z) - \xi(z')$ which is $\xi - (\xi + \xi(B_k))$.

Likewise, we explain the integrand $\{\xi - (\xi - \xi(A_k))\} d\eta$ along $B_k$ as follows. The integral comes from the sum of the integral over $B_k$ and over $-B_k$ and from grouping together the value $\xi(z)$ at a point $z$ of $B_k$ and the value $\xi(z')$ at the corresponding $z'$ of $-B_k$. We have $\xi(z') = \xi(z) - \xi(A_k)$, because we can compare $\xi(z')$ with $\xi(z)$ by going along $\partial\Omega$ in the counter-clockwise sense from the point $x$ of $B_k$ along $B_k$ to the juncture of $B_k$ and $-A_k$ and then along $-A_k$ to the juncture of $-A_k$ and $-B_k$ and then along $-B_k$ to $x'$, with the contribution from along $B_k$ cancelling the contribution from along $-B_k$ precisely. Since $B_k$ is in the positive sense and $-B_k$ has the opposite sense, so we end up with $\xi(z) - \xi(z')$ which is $\xi - (\xi - \xi(A_k))$.

In matrix notations we can write

$$\int_{\partial\Omega} \xi d\eta = \sum_{k=1}^{g} \left( \xi(A_k)\eta(B_k) - \xi(B_k)\eta(A_k) \right) = [\xi]' \, J \, [\eta].$$

By setting $\xi = \varphi$ and $\eta = w$, we get

$$\frac{1}{2\pi i}\int_C \varphi dw = \frac{1}{2\pi i}\,[\varphi]' J[w].$$

Since $[\varphi]$ is real by the construction of $dE_\ell(P)$ and $dF_\ell(P)$, it follows that

$$\mathrm{Re}\left(\frac{1}{2\pi i}\int_C \varphi dw\right) = \left(\frac{1}{2\pi}\,[\varphi]' J\right)\mathrm{Im}[w].$$

*Relation Between Contour Integrals and Residues.* Now if we take an $\mathbb{R}$-basis $dw_1, \cdots, dw_{2g}$ of holomorphic forms on $M$, then the imaginary part of the $2g \times 2g$ matrix $([w_1] \cdots [w_{2g}])$ of periods is nonsingular, otherwise by integrating the imaginary part of a real linear combination of $dw_1, \cdots, dw_{2g}$ we get a contradiction to the maximum principle for harmonic functions. This means that the set of all $\mathrm{Im}[w]$ with $dw$ holomorphic spans $\mathbb{R}^{2g}$. So $[\varphi]$ vanishes if and only if

$$\mathrm{Re}\left(\frac{1}{2\pi i}\int_C \varphi dw\right)$$

vanishes for all holomorphic forms $dw$.

This step simply means that when we later draw conclusions about periods from residues by the theory of residues, we have $2g$ independent such relations. That is the reason why we choose $dw$ in turn equal to $dw_1, \cdots, dw_{2g}$ and then use the nonsingularity of the matrix.

We compute

$$\mathrm{Re}\left(\frac{1}{2\pi i}\int_C \varphi dw\right)$$

by the residue formula. For the convenience of enumeration denote

$$F_\ell(P_k), \quad E_\ell(P_k) \quad (1 \leq \ell \leq \ell_k, 1 \leq k \leq n)$$

by $e_1, \cdots, e_{2m}$ and denote

$$\lambda_{k\ell}, \quad \mu_{k\ell} \quad (1 \leq \ell \leq \ell_k, 1 \leq k \leq n)$$

by $\lambda_1, \cdots, \lambda_{2m}$. Let $r_{k\ell}$ be the real part of the residue of $e_k dw_\ell$ at its only pole ($1 \leq k \leq 2m$, $1 \leq \ell \leq 2g$). By the theorem of residues

$$\mathrm{Re}\left(\frac{1}{2\pi i}\int_C \varphi dw_\ell\right) = \sum_{k=1}^{2m} \lambda_k r_{k\ell}.$$

Let $R$ denote the $2m \times 2g$ matrix $(r_{k\ell})$ and $\lambda$ denote the column $2m$-vector with entries $\lambda_k$. Then the vanishing of

$$\operatorname{Re}\left(\frac{1}{2\pi i}\int_C \varphi dw_\ell\right)$$

for all $1 \leq \ell \leq 2g$ simply means $\lambda' R = 0$. Thus $[\varphi] = 0$ if and only $\lambda' R = 0$. Let $r$ be the rank of $R$. Then $2a = 2m - r + 2$, where the term 2 comes from the *complex* constant of integration.

On the other hand we consider the equation $R\mu = 0$, where $\mu$ denotes a column $2g$-vector with real entries $\mu_\ell$. For $dw = \sum_{\ell=1}^{2g} \mu_\ell dw_\ell$, this equation means that the real part of the residue of $e_k dw$ at its only pole vanishes for all $1 \leq k \leq 2m$. When we calculate this explicitly by using the principal part of $dE_\ell(P_k)$, $dF_\ell(P_k)$ we conclude that $R\mu = 0$ if and only if $\operatorname{div} dw \geq D$. Hence $2b = 2g - r$.

*Final Step of the Proof of the Theorem of Riemann-Roch from Row-Rank Equal to Column-Rank for a Matrix.* Eliminating $r$ from the two equations $2a = 2m - r + 2$ and $2b = 2g - r$, we obtain the theorem of Riemann-Roch that $a - b = m - g + 1$ for the case of the nonnegative divisor $D$.

§4. *The canonical divisor.*

As an application of the theorem of Riemann-Roch we compute the degree of the divisor of a meromorphic form on a compact Riemann surface. First we observe that the degree is independent of the form chosen, because when we have two meromorphic forms $\alpha$ and $\beta$, their quotient is a meromorphic function and the degree of the divisor of any meromorphic function is zero. So to compute the degree of the divisor of a meromorphic form we take a holomorphic form $\alpha$ on the Riemann surface (we exclude the trivial case of genus zero which can easily be done directly). Let $D$ be the divisor of $\alpha$. The complex dimension $b$ of the space of all meromorphic forms $\beta$ with $\operatorname{div} \beta \geq D$ is 1, because $\frac{\beta}{\alpha}$ is a holomorphic function on the Riemann surface and must be a constant. By the theorem of Riemann-Roch we have $a = m - g + 2$, where $m$ is the degree of $D$ and $a$ is the complex dimension of the space $\mathcal{A}$ of all meromorphic functions $f$ with $\operatorname{div} f \geq -D$. The condition that $\operatorname{div} f \geq -D$ is equivalent to the holomorphicity of $f\alpha$. So multiplication by $\alpha$ gives us a one-one correspondence between $\mathcal{A}$ and the space of all holomorphic forms. Thus $a = g$ and from $a = m - g + 2$ we obtain $m = 2g - 2$. The divisor of a meromorphic form is called a *canonical divisor*.

§5. *The proof of the theorem of Riemann-Roch for general divisor.*

Now we prove the theorem of Riemann-Roch for the case of a general divisor. Take a divisor

$$D=\sum_{\mu=1}^{p}k_\mu P_\mu+\sum_{\nu=1}^{q}\ell_\nu Q_\nu.$$

where $k_\mu$ is positive and $\ell_\nu$ is negative. Let

$$m=\sum_{\mu=1}^{p}k_\mu,\quad n=\sum_{\nu=1}^{q}\ell_\nu.$$

Recall that $dE(P,Q)$ (respectively $dF(P,Q)$) is the meromorphic form on the compact Riemann surface whose principal part at $P$ is $\frac{dz}{z}$ and at $Q$ is $-\frac{dw}{w}$ (respectively at $P$ is $i\frac{dz}{z}$ and at $Q$ is $-i\frac{dw}{w}$) in some holomorphic local coordinate system $z$ centered at $P$ and in some holomorphic local coordinate system $w$ centered at $Q$. We will not need any information about the periods of $dE(P,Q)$ and $dF(P,Q)$ in the proof of the theorem of Riemann-Roch for a general divisor.

A general multi-valued meromorphic function with poles of order no more than $k_\mu$ at $P_\mu$ is

$$\varphi=\sum_{\mu=1}^{p}\sum_{k=1}^{k_\mu}\left(\alpha_{\mu k}E_k(P_\mu)+\beta_{\mu k}F_k(P_\mu)\right),$$

because when integrate

$$d\varphi=\sum_{\mu=1}^{p}\sum_{k=1}^{k_\mu}\left(\alpha_{\mu k}dE_k(P_\mu)+\beta_{\mu k}dF_k(P_\mu)\right),$$

we do not have to worry about residues at the poles of $dE_k(P_\mu)$ and $dF_k(P_\mu)$ which are all of order at least 2.

We require that $\varphi(Q_1)=0$ to eliminate the need for adding a constant of integration. The period $[\varphi]$ of $\varphi$ is real. We have to consider the condition that $\varphi$ vanishes to order at least $\ell_\nu$ at $Q_\nu$. Since $\varphi(Q_1)$ is zero already, the vanishing of $\varphi$ at $Q_1,\cdots,Q_q$ is equivalent to the vanishing of

$$\mathrm{Re}\left(\mathrm{Res}_{Q_\nu}\left(\varphi\ dE\left(Q_1,Q_\nu\right)\right)\right)\quad\text{and}\quad\mathrm{Re}\left(\mathrm{Res}_{Q_\nu}\left(\varphi\ dF\left(Q_1,Q_\nu\right)\right)\right)$$

for $2 \leq \nu \leq q$, because if $z$ is a local coordinate centered at $Q_\nu$ for some $2 \leq \nu \leq q$, then

$$\begin{cases} \varphi = \sum_{j=0}^{\infty} \gamma_j z^j, \\ dE\left(Q_1, Q_\nu\right) = \frac{dz}{z} + \text{ local holomorphic form,} \\ dF\left(Q_1, Q_\nu\right) = \frac{i\,dz}{z} + \text{ local holomorphic form.} \end{cases}$$

If $\varphi$ vanishes at $Q_1, \cdots, Q_q$, then $\varphi$ vanishes at $Q_\nu$ to order $\ell_\nu$ if and only if

$$\operatorname{Re}\left(\operatorname{Res}_{Q_\nu}\left(\varphi\ dE_\ell(Q_\nu)\right)\right) \quad \text{and} \quad \operatorname{Re}\left(\operatorname{Res}_{Q_\nu}\left(\varphi\ dF_\ell(Q_\nu)\right)\right)$$

vanish for $1 \leq \ell \leq \ell_\nu$, because if $z$ is a suitable local coordinate centered at $Q_\nu$ for some $2 \leq \nu \leq q$, then

$$\begin{cases} \varphi = \sum_{j=0}^{\infty} \gamma_j z^j, \\ dE_\ell\left(Q_\nu\right) = \frac{dz}{z^{\ell+1}} + \text{ local holomorphic form,} \\ dF_\ell\left(Q_\nu\right) = \frac{i\,dz}{z^{\ell+1}} + \text{ local holomorphic form.} \end{cases}$$

Let $dw_1, \cdots, dw_{2g}$ denote an $\mathbb{R}$-basis of holomorphic forms on $M$ and let $dw_{2g+1}, \cdots, dw_{2g+2n-2}$ be the meromorphic forms

$$dE(Q_1, Q_\lambda),\ dF(Q_1, Q_\lambda) \ \ (2 \leq \lambda \leq q),$$
$$dE_\ell(Q_\nu),\ dF_\ell(Q_\nu) \ \ (1 \leq \ell \leq \ell_\nu, 1 \leq \nu \leq q).$$

Let $e_1, \cdots, e_{2m}$ be the multi-valued functions

$$F_k(P_\mu),\ E_k(P_\mu) \ \ (1 \leq k \leq k_\mu,\ 1 \leq \mu \leq p).$$

Let

$$r_{\sigma\tau} = \operatorname{Re}\left(\sum_{\mu=1}^{p} \operatorname{Res}_{P_\mu}\left(e_\sigma dw_\tau\right)\right) \quad (1 \leq \sigma \leq 2m,\ 1 \leq \tau \leq 2g + 2n - 2).$$

(Note that though the sum of residues at all $P_1, \cdots, P_1$ is used, the sum only means a single term at the only pole of $e_\sigma$, because each $e_\sigma$ for $1 \leq \sigma \leq 2m$ has only a single pole. So this formulation is the same as the one used in the preceding case of nonnegative divisor. The reason for the use of a different notation is that later we will also consider $\sum_{\nu=1}^{q}$ of the residues at $Q_\nu$ for meromorphic forms $dE\left(Q_1, Q_\nu\right)$ and $dF\left(Q_1, Q_\nu\right)$, each with poles at more than one single point.) Since

$$\operatorname{Re}\left(\frac{1}{2\pi i}\int_C \varphi dw\right) = \left(\frac{1}{2\pi}\,[\varphi]' J\right)\operatorname{Im}[w]$$

for any meromorphic form $dw$, the period $[\varphi]$ vanishes if and only if

$$\operatorname{Re}\left(\frac{1}{2\pi i}\int_C \varphi dw\right)$$

vanishes for all holomorphic forms $dw$. Moreover, when this is the case the sum of all the residues of $\varphi dw$ is zero for any meromorphic form $dw$, because now $\varphi dw$ defines a global meromorphic form on the compact Riemann surface. If

$$\operatorname{Re}\left(\sum_{\mu=1}^{p}\operatorname{Res}_{P_\mu}\left(\varphi dw_\tau\right)\right)$$

is zero for $2g \le \tau \le 2g+2n-2$, then

$$\operatorname{Re}\left(\sum_{\nu=1}^{q}\operatorname{Res}_{Q_\nu}\left(\varphi dw_\tau\right)\right)$$

is zero for $2g \le \tau \le 2g+2n-2$ (because for each $2g \le \tau \le 2g+2n-2$ the only poles of the meromorphic form $dw_\tau$ are at $P_1, \cdots, P_p, Q_1, \cdots, Q_q$) and $\varphi$ must vanish to order $\ell_\nu$ at $Q_\nu$ for $1 \le \nu \le q$ (because the pole-set of each $w_\tau$ contains no more than one element of $Q_1, \cdots, Q_q$ except in the case of simple poles at $Q_1$ and $Q_\lambda$ for $dE(Q_1, Q_\lambda)$, $dF(Q_1, Q_\lambda)$ in which case $\varphi(Q_1) = 0$ is used to handle the situation.). Conversely, if $\varphi$ vanishes to order $\ell_\nu$ at $Q_\nu$ for $1 \le \nu \le q$, then

$$\operatorname{Re}\left(\sum_{\nu=1}^{q}\operatorname{Res}_{Q_\nu}\left(\varphi\, dw_\tau\right)\right)$$

is zero for $2g \le \tau \le 2g+2n-2$. Let $R$ be the $2m$ by $2g+2n-2$ matrix $(r_{\sigma\tau})$. Then $\varphi$ defines an element of $\mathcal{A}$ if and only if $\lambda' R = 0$, where $\lambda$ is a column $(2m)$-vector (with transpose $\lambda'$) whose components are the elements of the set

$$\{\alpha_{\mu k}\}_{1\leq k\leq k_\mu,\,1\leq\mu\leq p}\ \cup\ \{\beta_{\nu\ell}\}_{1\leq\ell\leq\ell_\nu,\,1\leq\nu\leq q}\,.$$

Let $r$ be the rank of $R$. Then $2a = 2m - r$. Here we do not have to add 2 to the right-hand side because we have normalized $\varphi$ by requiring that $\varphi(Q_1) = 0$.

On the other hand, for $dw$ to be an $\mathbb{R}$-linear combination of

$$dw_1, \cdots, dw_{2g+2n-2},$$

the vanishing of

$$\mathrm{Re}\left(\sum_{\mu=1}^{p}\mathrm{Res}_{P_\mu}\left(e_\sigma dw\right)\right)$$

for $1 \leq \sigma \leq 2m$ means that $dw$ has poles of order at most $\ell_\nu$ at $Q_\nu$ and vanishes to order at least $k_\mu$ at $P_\mu$. Thus $2b = 2g + 2n - 2 - r$. Eliminating $r$ from the equations for $2a$ and $2b$, we get $a - b = (m - n) - g + 1$, which is the statement of the theorem of Riemann-Roch.

§6. *Abel's Theorem.*

Recall that for elliptic functions we have the following three fundamental properties for their zero-sets and pole-sets. Inside a fundamental parallelogram the number of zeroes equals the number of poles, the sum of residues is zero, and the sum of the coordinates of the zeroes equals the sum of the coordinates of the poles modulo periods. Conversely, when we have the first and third statement for two given finite point sets, we can construct by using the Weierstrass $\sigma$-functions an elliptic functions having the two sets as zero-set and pole-set. For a meromorphic function $f$ on a general Riemann surface $M$ the first statement holds as one can see by integrating $\frac{1}{2\pi i}d\ \log f$ over $M$. The corresponding statement for the second statement is the vanishing of the sum of residue of $f\ dw$ for any holomorphic form $dw$ on $M$. The corresponding statement for the third statement and its converse is the theorem of Abel which states the following.

If $M$ is a compact Riemann surface and $\{P_1, \cdots, P_k\}$ and $\{Q_1, \cdots, Q_k\}$ are two finite point sets with possible duplication of members in each set, then there exists a meromorphic function having $\{P_1, \cdots, P_k\}$ as its zero-set and $\{Q_1, \cdots, Q_k\}$ as its pole-set if and only if there exists a closed loop $\gamma$ in $M$ such that for some paths joining $P_\mu$ to $Q_\mu$ one has

$$\sum_{\mu=1}^{k} \int_{P_\mu}^{Q_\mu} dw = \int_\gamma dw$$

for any holomorphic form $dw$ on $M$.

For the "if" part we let

$$\varphi = \sum_{\mu=1}^{k} E(P_\mu, Q_\mu)$$

and want to get $f$ as $e^\varphi$. So we want the period $[\varphi]$ of $\varphi$ to have entries in $2\pi i\mathbb{Z}$. Assume that the loop

$$\gamma = \sum_{\mu=1}^{g} (m_\mu A_\mu + n_\mu B_\mu).$$

Since

$$\begin{aligned}
\frac{1}{2\pi i}[w]' J[\varphi] &= \frac{1}{2\pi i}\int_C w d\varphi = \sum \operatorname{Res}(w d\varphi) \\
&= \sum_{\mu=1}^{k} \left(w(P_\mu) - w(Q_\mu)\right) = \sum_{\mu=1}^{k} \int_{Q_\mu}^{P_\mu} dw \\
&= \int_\gamma dw = [w]'(m_1, \cdots, m_g, n_1, \cdots, n_g)',
\end{aligned}$$

it follows that

$$[\varphi] = 2\pi i J^{-1} (m_1, \cdots, m_g, n_1, \cdots, n_g)'$$

and every entry of $[\varphi]$ is in $2\pi i\mathbb{Z}$. Conversely, if every entry of $[\varphi]$ is in $2\pi i\mathbb{Z}$, then we write

$$[\varphi] = 2\pi i \; J^{-1} (m_1, \cdots, m_g, n_1, \cdots, n_g)'$$

and

$$\sum_{\mu=1}^{k} \int_{Q_\mu}^{P_\mu} dw = \sum_{\mu=1}^{k} (w(P_\mu) - w(Q_\mu)) = \sum \operatorname{Res}(w d\varphi)$$

$$= \frac{1}{2\pi i}\int_C w d\varphi = \frac{1}{2\pi i}\,[w]' J[\varphi]$$

$$= [w]'\,(m_1, \cdots, m_g, n_1, \cdots, n_g)' = \int_\gamma dw.$$

This concludes the proof of Abel's theorem.

*Remark.* The theorem of Riemann-Roch and Abel's theorem could be interpreted as answering the question: for which configuration of charges, dipoles, or multipoles on a compact Riemann surface of genus $\geq 1$ would the flux functions (whose level curves are the flux lines and which are the harmonic conjugates of the electrostatic potential functions) in the case of the theorem of Riemann-Roch, or their exponentiation after multiplication by $2\pi i$ in the case of Abel's theorem, be single-valued on the Riemann surface so that the flux lines are closed curves?