Theorem of Selberg on Positive Percentage of Zeroes of Riemann Zeta Function on Critical Line

The key step of the proof of the theorem of Hardy-Little on an infinite number of zeroes of the Riemann zeta function on the critical line is the introduction of the real-valued function

$$(*) \qquad Z(t) == -2\pi^{\frac{1}{4}} \frac{\Xi(t)}{\left(t^2+\frac{1}{4}\right)\left|\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\right|}$$

on the real line and the difference of the growth rates of the two integrals

$$\int_{t=T}^{2T} Z(t)dt = O\left(T^{\frac{7}{8}}\right), \qquad \int_{t=T}^{2T} |Z(t)|\, dt > AT.$$

Let $N(T)$ be the number of zeroes of the function $\zeta(s)$ in $\{0 \leq \sigma \leq 1,\ 0 < t \leq T\}$ and let $N_0(T)$ be the number of zeroes of the function $\zeta(s)$ in $\left\{\sigma = \frac{1}{2},\ 0 < t \leq T\right\}$. We know that

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1+\log 2\pi}{2\pi} T + O\left(\log T\right).$$

The theorem of Selberg shows that there exists some $A > 0$ such that

$$N_0\left(T\right) \geq A\, T \log T$$

so that a positive percentage of the zeroes of the Riemann zeta function on the strip $0 \leq \sigma \leq 1$ lie on the critical line $\sigma = \frac{1}{2}$.

The key point of Selberg's argument is the introduction of another function to replace $Z(s)$ which plays the same role. His function $F(s)$ is defined as follows. Let

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_{\nu=1}^{\infty} \frac{\alpha_\nu}{\nu^s} \quad \text{with } \alpha_1' = 1.$$

Choose $X > 0$ and δ. Define

$$\beta_\nu = \begin{cases} \alpha_\nu \left(1 - \frac{\log \nu}{\log X}\right) & \text{for } 1 \leq \nu < X, \\ 0 \quad \text{for } \nu \geq X \end{cases}$$

and

$$F(t) = \frac{1}{\sqrt{2\pi}} \frac{\Xi(t)}{t^2+\frac{1}{4}} \left|\phi\left(\frac{1}{2}+it\right)\right|^2 e^{\left(\frac{\pi}{4}-\frac{\delta}{2}\right)t},$$

where

$$\phi(s) = \sum_{\nu=1}^{\infty} \frac{\beta_\nu}{\nu^s}.$$

There are the following three integral inequalities for this function $F(t)$ when X is chosen to be δ^{-c} for some $0 < c \leq \frac{1}{8}$ (which we assume to be the case).

$$\int_{t=-\infty}^{\infty} \left(\int_{u=t}^{t+u} |F(u)| \, du \right)^2 dt = O\left(\frac{h^2 \log\left(\frac{1}{\delta}\right)}{\delta^{\frac{1}{2}} \log X} \right),$$

$$\int_{t=-\infty}^{\infty} \left| \int_{u=t}^{t+u} F(u) \, du \right|^2 dt = O\left(\frac{h}{\delta^{\frac{1}{2}} \log X} \right),$$

$$\int_{t=0}^{T} \int_{u=t}^{t+h} |F(u)| \, du > AhT^{\frac{3}{4}} \quad \text{if } \delta = \frac{1}{T},$$

where A is a generic symbol for a positive constant.

The final argument after the three estimates is as follows. Choose $h > 0$. Let E be the subset of $t \in (0, T)$ such that

$$(\dagger) \qquad \int_{t}^{t+h} |F(u)| \, du > \left| \int_{t}^{t+h} F(u) du \right|.$$

On each $(t,\, t+u)$ the function $F(u)$ changes sign and there must be at least one zero of $\zeta\left(\frac{1}{2} + iu\right)$ in the interval. Since both sides of $(\dagger)$ agree except in E, it follows that

$$\begin{aligned} \int_E \left(\int_t^{t+h} |F(u)| \, du \right) dt &\geq \int_E \left(\left| \int_t^{t+h} F(u) du \right| - \left| \int_t^{t+h} F(u) du \right| \right) dt \\ = \int_{t=0}^{T} \left(\int_t^{t+h} |F(u)| \, du \right) dt &\geq \int_E \left(\left| \int_t^{t+h} F(u) du \right| - \left| \int_t^{t+h} F(u) du \right| \right) dt \\ &> AhT^{\frac{3}{4}} - \int_{t=0}^{T} \left| \int_t^{t+h} F(u) du \right| dt, \end{aligned}$$

because of the third integral inequality. By Hölder's inequality the left-hand side is not greater than

$$\left(\int_E dt \right)^{\frac{1}{2}} \left(\int_E \left(\int_t^{t+h} |F(u)| \, du \right)^2 dt \right)^{\frac{1}{2}}$$

$$\le (m(E))^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \left(\int_t^{t+h} |F(u)|\, du \right)^2 dt \right)^{\frac{1}{2}}$$

$$\le A\,(m(E))^{\frac{1}{2}}\, hT^{\frac{1}{4}} \left(\frac{\log T}{\log X} \right)^{\frac{1}{2}} \quad \text{when } \delta = \frac{1}{T}$$

from the first integral inequality. By Hölder's inequality the second term on the right-hand side is no greater than

$$\left(\int_0^T dt \right)^{\frac{1}{2}} \left(\int_0^T \left| \int_t^{t+h} F(u)du \right|^2 dt \right)^{\frac{1}{2}} < \frac{Ah^{\frac{1}{2}} T^{\frac{3}{4}}}{\log^{\frac{1}{2}} X}$$

from the second integral inequality when $\delta = \frac{1}{T}$. Hence

$$(m(E))^{\frac{1}{2}} > A_1 T^{\frac{1}{2}} \left(\frac{\log X}{\log T} \right)^{\frac{1}{2}} - A_2 \frac{T^{\frac{1}{2}}}{h^{\frac{1}{2}} \log^{\frac{1}{2}} T},$$

where A_1 and A_2 are generic symbols for positive constants. Set $T = \frac{1}{\delta}$ so that $X = T^c$. Choose $a > 0$ and set

$$h = \frac{1}{a \log X} = \frac{1}{ac \log T}.$$

Then

$$(m(E))^{\frac{1}{2}} > A_1 c^{\frac{1}{2}} T^{\frac{1}{2}} - A_2\,(ac)^{\frac{1}{2}}\, T^{\frac{1}{2}}.$$

Taking a small enough, we get

$$m(E) > A_3 T.$$

Among the intervals $(nh, (n+1)h)$ in $(0, T)$ at least $\frac{A_3 T}{h} - 1$ of them contains a point of t of E which must contain a zero of $\zeta\left(\frac{1}{2} + it\right)$. If the interval $(nh, (n+1)h)$ contains a point t of E, then $(nh, (n+2)h)$ contains the interval $(t, t+h)$ with $t \in E$. Thus there are at least

$$\frac{1}{2} \left(\frac{A_3 T}{h} - 1 \right)$$

zeroes of $\zeta\left(\frac{1}{2} + it\right)$ in $(0, T)$. It follows from

$$h = \frac{1}{ac \log T}$$

that

$$N_0(T) > AT \log T.$$