

Theta Divisors

Meromorphic Functions on Elliptic Curves as Quotients of Products of Translates of Theta Functions. We continue our theme of evaluating integrals of rational functions of two variables which are related by a polynomial. In the case of genus > 1 there are g simple integrals $\int dw_j$ ($1 \leq j \leq g$) coming from the holomorphic 1-forms on the Riemann surface. So the problem is to explicitly express the map

$$P \rightarrow w(P) = \begin{pmatrix} \int_{P_0}^P dw_1 \\ \cdot \\ \cdot \\ \cdot \\ \int_{P_0}^P dw_g \end{pmatrix}.$$

This map maps the Riemann surface M to \mathbb{C}^g/L , where L is the $g \times 2g$ period matrix. In the case of genus 1, one produces the Weierstrass \mathfrak{P} -function on \mathbb{C}/L so that $\mathfrak{P}(w(z))$ agrees with the projection map from M to \mathbb{P}_1 defined by $(x, y) \mapsto x$ when the finite part of M is given by

$$y^2 = 4x^3 - g_2x - g_3.$$

Since the Weierstrass \mathfrak{P} -function is known, the projection map $(x, y) \rightarrow x$ are both explicitly known and the integral $w(z)$ is explicitly known. We are going to follow the same path in the case of genus > 1 . We take an arbitrary known meromorphic function f on M and then try to construct explicitly a meromorphic function g on \mathbb{C}^g/L so that $f(z) = g(w(z))$. This way we make $w(z)$ explicitly known.

So the question is how to construct meromorphic functions on \mathbb{C}^g/L . For the case of $g = 1$ we have both the additive way and the multiplicative way of constructing meromorphic functions on \mathbb{C}/L . For the additive way we use a finite number of poles and principal parts at those poles. The only requirement is that the sum of the residues of the principal parts should be zero. Once this condition is satisfied we use the Weierstrass \mathfrak{P} function and its derivative \mathfrak{P}' and their powers and translates to construct the meromorphic function. For the multiplicative approach we assign a finite number of poles and an equal number of zeros. The only requirement is that the sum of the coordinates of the poles equal the sum of the coordinates of the zeros modulo the periods. Then one uses the Weierstrass sigma functions to

construct the meromorphic function with prescribed poles and zeros. The Weierstrass sigma function has only a single zero and no poles, but it is not doubly periodic. Its logarithmic derivative is a Weierstrass zeta function whose derivative is doubly periodic. So the Weierstrass sigma function satisfies $\zeta(w + \omega) = \zeta(w) + c_\omega$ and the Weierstrass sigma function satisfies $\sigma(w + \omega) = \sigma(w) \exp(c_\omega w + d_\omega)$. One way to construct the Weierstrass sigma function is to start with a function with a simple zero and then form an infinite product of its translates by periods after modification by the Weierstrass factor to insure convergence. Another way to use infinite series as we have done in the construction of the Jacobian theta functions. Let us recall the construction of the Jacobian theta function by an infinite series. We use the method of undetermined coefficients to come up with the infinite series

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niw},$$

where $q = e^{\pi\tau i}$ and τ is the quotient $\frac{\omega_2}{\omega_1}$ of the two periods ω_1 and ω_2 . In this method of undetermined coefficients we simply look for a function with the simplest periodicity factors and we have no control over its zeros. Once we get two such functions with the same periodicity factors or up to a sign, we can construct meromorphic functions from them.

Definition of Theta Functions of Several Variables. Now in the higher dimensional case \mathbb{C}^g/L a pole set or a zero set is given by a hypersurface and it is difficult to find such hypersurfaces and even more difficult to find principal parts. So it is not possible to try to directly construct the analog of a Weierstrass \mathfrak{P} function by infinite series or the analog of the Weierstrass sigma function by infinite product. The only way left is to imitate the construction of the Jacobian theta function by infinite series, because in that construction no pole-set or zero-set is prescribed. Let us now look closer at the infinite series

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niw}.$$

We rewrite it as

$$\sum_{n=-\infty}^{\infty} \exp(n^2\pi\tau i + 2niw + n\pi i).$$

Let us now imitate it in this form in the higher dimensional case. Instead of the integer n we use the g -tuple of integers t with components $t_1, \dots, t_g \in \mathbb{Z}$

written as a column g -vector. Instead of $w \in \mathbb{C}$ we use the g -tuple of complex numbers s with components $s_1, \dots, s_g \in \mathbb{C}$ written as a column g -vector. Instead of τ in the upper half plane we use the matrix Z in the Siegel upper half space. So $Z = X + iY$ is a symmetric $g \times g$ matrix with complex entries such that $Y = \text{Im } Z > 0$. Instead of

$$\sum_{n=-\infty}^{\infty} \exp(n^2 \pi \tau i + 2niw + n\pi i)$$

we define

$$\Theta(s, Z) = \sum_{t \in \mathbb{Z}^g} \exp(\pi i(t' Z t + 2t' s)).$$

We also use the simpler notation $\Theta(s)$ to denote $\Theta(s, Z)$. The function $\Theta(s)$ is not identically zero. The reason is as follows. When

$$s = (e^{i\theta_1}, \dots, e^{i\theta_n}),$$

the function $\Theta(s)$ becomes a Fourier series in the real variables $(\theta_1, \dots, \theta_n)$. If the function $\Theta(s)$ is identically zero in s and in particular identically zero in the real variables $(\theta_1, \dots, \theta_n)$, then all its Fourier coefficients as a function of $(\theta_1, \dots, \theta_n)$ must be identically zero. However, for a positive definite matrix $\text{Im } Z$ there exists a real vector v such that $v'(\text{Im } Z)v$ is positive. The vector v is the limit of a sequence of vectors of the form $\frac{1}{p}t$ where p is a positive integer and $t \in \mathbb{Z}^n$. Thus $t'(\text{Im } Z)t$ is nonzero for some $t \in \mathbb{Z}^n$, contradicting the vanishing of all Fourier coefficients of

$$\Theta(s) = \sum_{t \in \mathbb{Z}^g} \exp(\pi i(t' Z t + 2t' s))$$

as a function of $(\theta_1, \dots, \theta_n)$. So we know that the function $\Theta(s)$ is not identically zero as a function of s . The function $\Theta(s)$ is even, because when s is replaced by $-s$, we can replace t by $-t$ and the sum remains unchanged. It is reduced to

$$\sum_{n=-\infty}^{\infty} \exp(n^2 \pi \tau i + 2niw + n\pi i)$$

when one sets $t = n$ and $s = \frac{w}{\pi} + \frac{1}{2}$.

Periodicity Factors. Let Z be a symmetric $g \times g$ matrix with complex coefficients whose imaginary part $\text{Im } Z$ is positive-definite. The period matrix is given by $\begin{pmatrix} E \\ Z \end{pmatrix}$ and the $2g$ periods are the column vectors of the identity matrix E and the matrix Z . We now compute the $2g$ periodicity factors for the theta function $\Theta(s)$ defined by Z . The periodicity factors are given by the following proposition.

Proposition (Periodicity of Theta Functions). The periodicity factor for $v \in \mathbb{Z}^g$ is 1 and the periodicity factor for Zv with $v \in \mathbb{Z}^g$ is $\exp(\pi i(-v'Zv - 2v's))$, where s is the variable of \mathbb{C}^g .

Proof. Let $Z = (z_{\mu\nu})_{1 \leq \mu, \nu \leq g}$. For $v \in \mathbb{Z}^g$ we have

$$\begin{aligned} \Theta(s+v) &= \sum_{t \in \mathbb{Z}^g} \exp(\pi i((t'Zt + 2t'(s+v)))) \\ &= \sum_{t \in \mathbb{Z}^g} \exp(\pi i(t'Zt + 2t's)) \exp(2\pi i t'v) \\ &= \sum_{t \in \mathbb{Z}^g} \exp(\pi i(t'Zt + 2t's)) = \Theta(s) \end{aligned}$$

and

$$\begin{aligned} \Theta(s+Zv) &= \sum_{t \in \mathbb{Z}^g} \exp(\pi i(t'Zt + 2t'(s+Zv))) \\ &= \sum_{t \in \mathbb{Z}^g} \exp(\pi i((t-v)'Z(t-v) + 2(t-v)'(s+Zv))) \\ &\quad \text{(after replacing the dummy index } t \text{ by } t-v) \\ &= \sum_{t \in \mathbb{Z}^g} \exp(\pi i(t'Zt - 2t'Zv + v'Zv + 2t's - 2v's + 2t'Zv - 2v'Zv)) \\ &= \sum_{t \in \mathbb{Z}^g} \exp(\pi i(t'Zt - v'Zv + 2t's - 2v's)) \\ &= \exp(\pi i(-v'Zv - 2v's)) \sum_{t \in \mathbb{Z}^g} \exp(\pi i(t'Zt + 2t's)) \\ &= \exp(\pi i(-v'Zv - 2v's)) \Theta(s). \end{aligned}$$

Q.E.D.

In the case of genus 1, we construct the Jacobian theta function ϑ_4 by infinite series and then by translation by half-periods get the other three

Jacobian theta functions $\vartheta_1, \vartheta_2, \vartheta_3$. The Jacobian elliptic functions $\operatorname{sn}, \operatorname{cn}, \operatorname{dn}$ are then obtained by taking the squares of the quotients of two appropriate Jacobian theta functions. So we can think of the Jacobian elliptic functions as product of quotients of translates of the Jacobian theta functions. Elliptic functions correspond in a one-one manner to meromorphic functions on the Riemann surface of genus 1 via the function $w = w(z)$ which is the indefinite integral of a holomorphic 1-form. Now in the case of genus > 1 , analogously we want to express a given meromorphic function f on the Riemann surface M as the composite of $w(z)$ and a product of quotients of translates of the theta function $\Theta(s, Z)$. To conclude that $f(z)$ agrees with the composite of $w(z)$ and a product of quotients of translates of the theta function $\Theta(s, Z)$, we use the usual argument that both have the same set of zeros and poles on M and therefore differ by a multiplicative constant only. So we should first know the zero set of $\Theta(w(z) - u)$. We first count the number of zeroes of $\Theta(w(z) - u)$ and then determine their locations in the Riemann surface.

Number of Zeroes of Pullback of Theta Function by Jacobian Map. We want to count by using the argument principle how many zeros the function $\Theta(w(z) - u)$ has. It has precisely g zeros if it is not identically zero.

To prove it, we let $\varphi(z) = \Theta(w(z) - u)$. The function φ is multi-valued. To make it single-valued, we use a fundamental polygon Ω . We choose the initial point P_0 of integration to be a vertex of Ω so that w vanishes at the vertices of Ω modulo the values of dw on a loop. To apply the argument principle to find the number of zeros of φ on Ω , we want to find the total change in $\frac{1}{2\pi i} \log \varphi$ along the boundary $\partial\Omega$ of Ω . To calculate the change along the sides $A_k, B_k, -A_k, -B_k$, we group together A_k and $-A_k$ and group together B_k and $-B_k$.

Let z be a point on A_k and z' be the corresponding point on $-A_k$. Then $w(z') - w(z) = w(B_k) = Z e_k$ where e_k is the column vector with 1 in the k -th place and 0 elsewhere. So

$$\begin{aligned} \varphi(z') &= \Theta(w(z') - u) = \Theta(w(z) - u + Z e_k) \\ &= \exp(\pi i (-e'_k Z e_k - 2e'_k(w(z) - u))) \Theta(w(z) - u) \\ &= \exp(\pi i (-e'_k Z e_k - 2e'_k(w(z) - u))) \varphi(z) \end{aligned}$$

and

$$\frac{1}{2\pi i} \log \varphi(z') = \frac{1}{2} (-e'_k Z e_k - 2e'_k(w(z) - u)) + \frac{1}{2\pi i} \log \varphi(z) \text{ modulo } \mathbb{Z}.$$

Let us use Δ_{A_k} and Δ_{-A_k} to denote the changes along A_k and $-A_k$ respectively, In other words, $\Delta_A \psi$ means the integral of $d\psi$ along A which is also the difference of the values of ψ at the final end-point and the initial end-point of A . We have

$$\begin{aligned} \Delta_{-A_k} \frac{1}{2\pi i} \log \varphi + \Delta_{A_k} \frac{1}{2\pi i} \log \varphi &= \Delta_{A_k} \frac{1}{2\pi i} \log \varphi(z) - \Delta_{A_k} \frac{1}{2\pi i} \log \varphi(z') \\ &= \Delta_{A_k} \frac{1}{2} (e'_k Z e_k + 2e'_k(w - u)) = \Delta_{A_k} e'_k w = 1, \end{aligned}$$

because in the expression $\frac{1}{2} (e'_k Z e_k + 2e'_k(w - u))$ only the function w is not a single-valued function on the Riemann surface M . Now let ζ be a point on B_k and ζ' be the corresponding point on $-B_k$. Then

$$w(\zeta') - w(\zeta) = -w(A_k) = -e_k$$

and

$$\varphi(\zeta') = \Theta(w(\zeta') - u) = \Theta(w(\zeta) - u - e_k) = \Theta(w(\zeta) - u) = \varphi(w(\zeta)).$$

Hence the change of $\frac{1}{2\pi i} \log \varphi$ along B_k cancels the change of $\frac{1}{2\pi i} \log \varphi$ along $-B_k$. The total change of $\frac{1}{2\pi i} \log \varphi$ along $\partial\Omega$ is g . This concludes the proof that φ has precisely g zeros.

Location of Zeroes of Pullback of Theta Function by Jacobian Map. Next we want to locate the g zeros of φ . These g zeros of φ will turn out to be the solution of the inversion problem. The g zeros of course will depend on the choice of the g -vector u . In order to put our answer concerning the g zeros of φ in a simple form, we write $u = s - c$, where c is a column g -vector whose k -th component c_k is given by

$$c_k = -\frac{1}{2} z_{kk} + \sum_{\ell=1}^g \int_{A_\ell} w_k dw_\ell$$

and the integration is along the side A_ℓ of the fundamental polygon Ω . In other words,

$$c = \begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_g \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} z_{11} \\ \cdot \\ \cdot \\ \cdot \\ z_{gg} \end{pmatrix} + \sum_{\ell=1}^g \int_{A_\ell} \begin{pmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_g \end{pmatrix} dw_\ell.$$

In this formula we assume that one of the vertices of Ω is the initial point of integration P_0 so that w is zero at the vertices modulo periods. Since $w_\ell(A_\ell) = 1$, when a different vertex is chosen as the initial point of integration, a curve γ in Ω becomes a loop in M and the new g -vector c with components c_1, \dots, c_g differs from the old one only by g times $w(\gamma)$. The answer concerning the location of the zeros of φ is as follows.

Theorem (Jacobi Inversion). Let $\varphi(z) = \Theta(w(z) - s + c)$ be the pullback, by the Jacobian map $z \mapsto w(z)$, of the Theta function Θ translated by $s - c$. Let P_1, \dots, P_g be the g zeros of φ . Then

$$\sum_{k=1}^g w(P_k) \equiv s,$$

where the congruence means modulo the period lattice. In particular, the g points precisely solve the Jacobi inversion problem

$$\sum_{k=1}^g \int_{z_0}^{P_k} dw_\ell = s_\ell \quad (1 \leq \ell \leq g).$$

Remark. Before we give a proof for this, let us look at the meaning of c in the case of genus one. We assume that the torus is $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ with $\text{Im } \tau > 0$. There

$$c_1 = -\frac{1}{2}z_{11} + \int_{A_1} w_1 dw_1.$$

Since $z_{11} = \tau$ and A_1 is the loop corresponding to the image of the interval $[0,1]$ in $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, it follows that

$$c_1 = -\frac{1}{2}\tau + \frac{1}{2}.$$

So in the case of $g = 1$ the zero of $\Theta(w + c)$ is at $w = 0$ (when we use $s = 0$). In other words, the zero of $\Theta(w)$ is at $w = c$. This agrees with our earlier discussion of the fourth Jacobi theta function which has a zero at $\frac{\tau}{2}$. The reason for adding $\frac{1}{2}$ is the transformation $s = \frac{w}{\pi} + \frac{1}{2}$ at the time when we define the function Θ .

Proof of Theorem on Jacobi Inversion. By the residue theorem we have

$$\frac{1}{2\pi i} \int_{\partial\Omega} w d \log \varphi = \sum_{k=1}^g w(P_k).$$

Again in the computation of the integral over $\partial\Omega$ we group together A_k and $-A_k$ and group together B_k and $-B_k$. We keep the notations which were introduced in the earlier argument of determining the number of zeros of φ . Let z be a point on A_k and z' be the corresponding point on $-A_k$. Since

$$\frac{1}{2\pi i} \log \varphi(z') = \frac{1}{2} (-e'_k Z e_k - 2e'_k(w(z) - u)) + \frac{1}{2\pi i} \log \varphi(z) \pmod{\mathbb{Z}},$$

it follows that

$$d \log \varphi(z') = -2\pi i dw_k(z) + d \log \varphi(z)$$

and

$$\begin{aligned} & \frac{1}{2\pi i} \int_{A_k} w(z) d \log \varphi(z) - \frac{1}{2\pi i} \int_{A_k} w(z') d \log \varphi(z') \\ = & \frac{1}{2\pi i} \int_{A_k} w(z) d \log \varphi(z) - \frac{1}{2\pi i} \int_{A_k} (w(z) + w(B_k)) (-2\pi i dw_k(z) + d \log \varphi(z)) \\ = & \int_{A_k} w(z) dw_k(z) + w(B_k) \int_{A_k} dw_k(z) - w(B_k) \frac{1}{2\pi i} \int_{A_k} d \log \varphi(z) \\ = & \int_{A_k} w(z) dw_k(z) + w(B_k) w_k(A_k) - w(B_k) \frac{1}{2\pi i} \Delta_{A_k} \log \varphi(z). \end{aligned}$$

Since $w(A_k) = 1$ and

$$\frac{1}{2\pi i} \Delta_{A_k} \log \varphi(z)$$

is an integer (due to the fact that $\varphi(\zeta)$ assumes the same value at $\zeta \in B_k$ and at the corresponding point $\zeta' \in -B_k$ and as a consequence $\log \varphi(\zeta')$ and $\log \varphi(\zeta)$ differ by an integer times 2π), it follows that

$$\frac{1}{2\pi i} \int_{A_k} w(z) d \log \varphi(w(z)) - \frac{1}{2\pi i} \int_{A_k} w(z') d \log \varphi(w(z')) \equiv \int_{A_k} w(z) dw_k(z)$$

modulo the period lattice. Now let ζ be a point on B_k and ζ' be the corresponding point on $-B_k$. Then

$$\begin{aligned} & \frac{1}{2\pi i} \int_{B_k} w(\zeta) d \log \varphi(\zeta) - \frac{1}{2\pi i} \int_{B_k} w(\zeta') d \log \varphi(\zeta') \\ = & \frac{1}{2\pi i} \int_{B_k} (w(\zeta) - w(\zeta')) d \log \varphi(\zeta) \end{aligned}$$

$$= \frac{1}{2\pi i} w(A_k) \int_{B_k} d \log \varphi(\zeta) = w(A_k) \frac{1}{2\pi i} \Delta_{B_k} \log \varphi(\zeta).$$

Here we cannot say that

$$\frac{1}{2\pi i} \Delta_{B_k} \log \varphi(\zeta)$$

is an integer, because the value of $\varphi(z)$ at a the intersection point z of A_k and B_k is not the same as the value at the intersection point z' of $-A_k$ and B_k . Since $\Delta_{B_k} \log \varphi(\zeta)$ is the difference of the values of $\log \varphi$ at the two end-points of B_k , it follows that

$$\Delta_{B_k} \log \varphi(\zeta) = \log \varphi(z') - \log \varphi(z).$$

From

$$\frac{1}{2\pi i} \log \varphi(z') = \frac{1}{2} (-e'_k Z e_k - 2e'_k(w(z) - u)) + \frac{1}{2\pi i} \log \varphi(z) \pmod{\mathbb{Z}}$$

and the vanishing of w at the vertices of Ω modulo the values of dw along some loop (so that

$$\frac{1}{2} (-2e'_k) w(z) = w_k(z)$$

with the value of w_k at the vertex z being a period), it follows that

$$\frac{1}{2\pi i} \Delta_{B_k} \log \varphi(\zeta)$$

is equal to

$$\frac{1}{2} (-e'_k Z e_k + 2e'_k u) = -\frac{1}{2} z_{kk} + u_k + w_k(\gamma)$$

(where γ is some loop) and

$$\begin{aligned} & \frac{1}{2\pi i} \int_{B_k} w(\zeta) d \log \varphi(\zeta) - \frac{1}{2\pi i} \int_{B_k} w(\zeta') d \log \varphi(\zeta') \\ &= -w(A_k) \frac{1}{2} z_{kk} + u_k w(A_k) + w_k(\gamma) w(A_k). \end{aligned}$$

Note that here we replace the congruence modulo \mathbb{Z} by incorporating into the loop γ some integral combination of A_ℓ , because the integral of dw_k over A_ℓ is the Kronecker delta $\delta_{k\ell}$. Finally

$$\frac{1}{2\pi i} \int_{\partial\Omega} w d \log \varphi$$

$$\begin{aligned} &\equiv \sum_{k=1}^g \left(\int_{A_k} w(z) dw_k(z) - w(A_k) \frac{1}{2} z_{kk} + u_k w(A_k) + w_k(\gamma) w(A_k) \right) \\ &= c + u + w(\gamma) \equiv s. \end{aligned}$$

This finishes the proof of the theorem on Jacobi inversion. Q.E.D.

Theta Divisor as Image of Symmetric Product of $g - 1$ Riemann Surface.

Recall that

$$c = \begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_g \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} z_{11} \\ \cdot \\ \cdot \\ \cdot \\ z_{gg} \end{pmatrix} + \sum_{\ell=1}^g \int_{A_\ell} \begin{pmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_g \end{pmatrix} dw_\ell.$$

Let us reformulate our result of the Jacobi inversion problem in the following theorem. The relation will be clear from the proof of the “if” part.

Theorem. $\Theta(t - c) = 0$ if and only if $t = \sum_{\ell=2}^g w(y_\ell)$. In other words, after we modify by the offset c the zeroes of the theta function are precisely in the image of $g - 1$ copies of M .

Proof. The “if” part. Assume first that y_2, \dots, y_g are generic. Generically pick another y_1 . Let $s \equiv \sum_{\ell=1}^g w(y_\ell)$. Consider $\Theta(w(z) - s + c)$. Let its zeroes be z_1, \dots, z_g . We know that $\sum_{\ell=1}^g w(z_\ell) \equiv s$. We want to conclude from

$$\sum_{\ell=1}^g w(y_\ell) \equiv \sum_{\ell=1}^g w(z_\ell)$$

that

$$\{y_1, \dots, y_g\} = \{z_1, \dots, z_g\}.$$

Suppose we can do that. In particular, y_1 is in the set $\{z_1, \dots, z_g\}$ and $\Theta(w(y_1) - s + c) = 0$. That means

$$\Theta \left(w(y_1) - \sum_{\ell=1}^g w(y_\ell) + c \right) = 0$$

and

$$\Theta \left(-\sum_{\ell=2}^g w(y_\ell) + c \right) = 0.$$

Since the theta function Θ is even, we have

$$\Theta \left(\sum_{\ell=2}^g w(y_\ell) - c \right) = 0.$$

When y_2, \dots, y_g are not generic, we approximate the set by a generic one and pass to limit. Now we want to discuss how one can conclude that generically

$$\{y_1, \dots, y_g\} = \{z_1, \dots, z_g\}$$

follows from

$$\sum_{\ell=1}^g w(y_\ell) \equiv \sum_{\ell=1}^g w(z_\ell).$$

In other words, we want to show that generically the map from symmetric product of g copies of M to \mathbb{C}^g/L is injective. Its proof makes use of the theorem of Riemann-Roch and Abel's theorem and the introduction of the concept of general divisors. These are divisors of degree g or $g - 1$ at whose support the map from the product of g or $g - 1$ copies of M to \mathbb{C}^g/L is injective. Let us interrupt our proof to discuss the general divisors.

General and Complementary Divisors.

Definition. A divisor $D = \sum_{\ell=2}^g P_\ell$ is *general* if there exists no nonconstant meromorphic function f with $\text{div } f \geq -D$.

Another way to formulate is as follows. By the theorem of Riemann-Roch, since $\dim_{\mathbb{C}} \mathcal{A}_D = 1$ and $\dim_{\mathbb{C}} \mathcal{A}_D - \dim_{\mathbb{C}} \mathcal{B}_D = \deg D + 1 - g = 0$, we have a unique holomorphic form ω (up to a multiplicative constant) with $\text{div } \omega \geq D$. (Recall the notation that \mathcal{A}_D is the set of all meromorphic functions on the Riemann surface whose divisors are $\geq -D$ and \mathcal{B}_D is the set of all meromorphic forms on the Riemann surface whose divisors are $\geq D$.) So a divisor D of degree $g - 1$ is general if and only if there exists a unique holomorphic form ω (up to a multiplicative constant) with $\text{div } \omega \geq D$.

Since every holomorphic form has precisely $2g - 2$ zeroes, we have the other zeroes Q_2, \dots, Q_g . The divisor

$$D' = \sum_{\ell=2}^g Q_\ell$$

is called the *complementary divisor* of D . The complementary divisor D' is also general. Otherwise there exists a nonconstant meromorphic function f with $\operatorname{div} f \geq -D'$ and $\operatorname{div}(f\omega) \geq -D' + \operatorname{div} \omega = D$, contradicting the fact that there exists a unique holomorphic form ω (up to a multiplicative constant) with $\operatorname{div} \omega \geq D$.

Another characterization of a general divisor of degree $g - 1$ is as follows. A divisor $D = \sum_{\ell=2}^g P_\ell$ of degree $g - 1$ is general if and only if for any divisor $D' = \sum_{\ell=2}^g P'_\ell$ with

$$\sum_{\ell=2}^g w(P_\ell) \equiv \sum_{\ell=2}^g w(P'_\ell)$$

we have $D = D'$. In other words the map from the product of $g - 1$ copies of M to \mathbb{C}^g/L is injective at the points of a general divisor. The reason is as follows. By Abel's theorem, we know that

$$\sum_{\ell=2}^g w(P_\ell) \equiv \sum_{\ell=2}^g w(P'_\ell)$$

is equivalent to the existence of a meromorphic function F with zeroes at P'_2, \dots, P'_g and poles at P_2, \dots, P_g and the existence of such a meromorphic function F is the same as the existence of a nonconstant meromorphic function f with $\operatorname{div} f \geq -D$, because we can make the pole set of f equal to $\{P_2, \dots, P_g\}$ by enlarging the pole set and the zero set at the same time.

Most divisors $D = \sum_{\ell=2}^g P_\ell$ of degree $g - 1$ are general. To see this we use the characterization of a general divisor D by $\dim_{\mathbb{C}} \mathcal{B}_D = 1$. An element ω of \mathcal{B}_D is a linear combination $\sum_{\ell=1}^g a_\ell dw_\ell$ of dw_1, \dots, dw_g . The vanishing of ω at P_2, \dots, P_g means

$$(*) \quad \begin{pmatrix} dw_1(P_2) & \cdots & dw_g(P_2) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ dw_1(P_g) & \cdots & dw_g(P_g) \end{pmatrix} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_g \end{pmatrix} = 0.$$

By induction on the number of points and expansion of the determinant according to the last row, we know that generically we can choose P_2, \dots, P_g

such that the rank of the matrix

$$\begin{pmatrix} dw_1(P_2) & \cdots & dw_g(P_2) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ dw_1(P_g) & \cdots & dw_g(P_g) \end{pmatrix}$$

is $g - 1$ and so the solution space of the equation (*) has precisely complex dimension one.

Let $D = \sum_{\ell=1}^g P_\ell$ be a divisor of degree g . We say that D is *general* if and only if $\sum_{\ell=2}^g P_\ell$ is general and P_1 is not in the support of the complementary divisor D' of $\sum_{\ell=2}^g P_\ell$. Another characterization is that D is general if and only if there exists no holomorphic form ω with $\text{div } \omega \geq D$. The reason is that if there exists uniquely ω (up to a multiplicative constant) with $\text{div } \omega \geq -\sum_{\ell=2}^g P_\ell$, then whether ω vanishes also at P_1 depends on whether P_1 belongs to the complementary divisor of D . By the theorem of Riemann-Roch

$$\dim_{\mathbb{C}} \mathcal{A}_D - \dim_{\mathbb{C}} \mathcal{B}_D = \deg D + 1 - g = 1.$$

Hence the vanishing of $\dim_{\mathbb{C}} \mathcal{B}_D$ means that $\dim_{\mathbb{C}} \mathcal{A}_D = 1$. Only the constants functions f would satisfy $\text{div } f \geq -D$. In other words, again the map from the products of g copies of M to \mathbb{C}^g/L is precisely injective at the points of a general divisor of degree g . By the same argument about the rank of

$$\begin{pmatrix} dw_1(P_1) & \cdots & dw_g(P_1) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ dw_1(P_g) & \cdots & dw_g(P_g) \end{pmatrix}$$

we know that a generic divisor of degree g is general.

We now get back to the proof of the theorem. By the use of generic divisors we conclude that generically

$$\{y_1, \cdots, y_g\} = \{z_1, \cdots, z_g\}$$

follows from

$$\sum_{\ell=1}^g w(y_\ell) \equiv \sum_{\ell=1}^g w(z_\ell).$$

Thus the proof of the “if” part is complete.

For the “only if” part, we suppose that $\Theta(t - c) = 0$ for some t . Consider

$$\varphi(P_1) = \Theta \left(w(P_1) + \sum_{k=2}^m w(P_k) - \sum_{k=1}^m w(Q_k) - t + c \right).$$

Let m be the smallest integer such that there exist

$$P_1, \dots, P_m, Q_1, \dots, Q_m$$

with $\varphi(P_1) \neq 0$. Note that when $m = 0$, $\varphi(P_1)$ becomes $\Theta(-t + c) = \Theta(t - c) = 0$. The function $\varphi(P_1)$ has precisely g zeros. By the minimality of m ,

$$\begin{aligned} \varphi(Q_\ell) &= \Theta \left(w(Q_\ell) + \sum_{k=2}^m w(P_k) - \sum_{k=1}^m w(Q_k) - t + c \right) \\ &= \Theta \left(\sum_{k=2}^m w(P_k) - \sum_{k=1, k \neq \ell}^m w(Q_k) - t + c \right) \\ &= \Theta \left(w(P_m) + \sum_{k=2}^{m-1} w(P_k) - \sum_{k=1, k \neq \ell}^m w(Q_k) - t + c \right) = 0 \end{aligned}$$

for $1 \leq \ell \leq m$. Since the nonvanishing of $\varphi(P_1)$ is an open condition, we can assume without loss of generality that Q_1, \dots, Q_m are distinct. The the first m zeros of $\varphi(P_1)$ as a function of P_1 are Q_1, \dots, Q_ℓ . Let the other zeros be Q'_{m+1}, \dots, Q'_g . Then

$$s \equiv \sum_{k=1}^m w(Q_k) + \sum_{k=m+1}^g w(Q'_k),$$

where

$$s = t - \sum_{k=2}^m w(P_k) + \sum_{k=1}^m w(Q_k).$$

Hence

$$t \equiv \sum_{k=2}^m w(P_k) + \sum_{k=m+1}^g w(Q'_k).$$

This concludes the proof of the “only if” part.

Constructing Theta Functions Whose Pullbacks Have Prescribed Zeros.

Theorem. Let $x_2 + \cdots + x_g$ be a general divisor and $y_2 + \cdots + y_g$ be its complementary divisor. Let $Q \neq x_2, \cdots, x_g$. Then

$$\varphi(P, Q) = \Theta \left(w(P) - w(Q) - c + \sum_{k=2}^g w(x_k) \right)$$

as a function of P has precisely g zeroes Q, y_2, \cdots, y_g .

Proof. Let P be a zero of $\varphi(P, Q)$. We would like to prove that P is one of Q, y_2, \cdots, y_g . Suppose P is not one of y_2, \cdots, y_g . We have to show that P is equal to Q . Since P is a zero of $\varphi(P, Q)$, there exist R_2, \cdots, R_g such that

$$w(P) - w(Q) + \sum_{k=2}^g w(x_k) \equiv \sum_{k=2}^g w(R_k).$$

We are going to denote this congruence by

$$P + \sum_{k=2}^g x_k \approx Q + \sum_{k=2}^g R_k.$$

Since $\sum_{k=2}^g x_k$ is a general divisor and since P is not one of y_2, \cdots, y_g , we know that $P + \sum_{k=2}^g x_k$ is a general divisor and it follows from

$$P + \sum_{k=2}^g x_k \approx Q + \sum_{k=2}^g R_k$$

that

$$P + \sum_{k=2}^g x_k = Q + \sum_{k=2}^g R_k.$$

Since Q is not equal to any of x_2, \cdots, x_g , from this equality we conclude that $Q = P$.

Now we verify that Q, y_2, \cdots, y_g are roots of the function $\varphi(P, Q)$ of P . Clearly Q is a root of the function $\varphi(P, Q)$ of P , because

$$\varphi(Q, Q) = \Theta \left(w(Q) - w(Q) - c + \sum_{k=2}^g w(x_k) \right)$$

$$= \Theta \left(-c + \sum_{k=2}^g w(x_k) \right) = 0.$$

To verify that y_k ($2 \leq k \leq g$) is a root of the function $\varphi(P, Q)$ of P , we can assume without loss of generality that $k = 2$. Then $y_2 + \sum_{k=2}^g x_k$ is a special divisor. So we can find a non-identically-zero meromorphic function f with $\text{div } f \geq -y_2 - \sum_{k=2}^g x_k$ and $f(Q) = 0$. So

$$y_2 + \sum_{k=2}^g x_k \approx Q + \sum_{k=2}^g R_k$$

for some R_2, \dots, R_g . Thus

$$\begin{aligned} \varphi(y_2, Q) &= \Theta \left(w(y_2) - w(Q) - c + \sum_{k=2}^g w(x_k) \right) \\ &= \Theta \left(-c + \sum_{k=2}^g w(R_k) \right) = 0. \end{aligned}$$

The only thing left to be checked is the question of multiplicity when some of the points y_2, \dots, y_g are equal. Let z_1, \dots, z_g be the g zeroes of the function $\varphi(P, Q)$ of P with multiplicity counted. We can assume that $z_1 = Q$. Then

$$(*) \quad \sum_{k=1}^g w(z_k) \equiv w(Q) + 2c - \sum_{k=2}^g w(x_k),$$

because

$$\sum_{\ell=1}^g w(z_\ell) \equiv s$$

for the roots z_1, \dots, z_g of the function $\Theta(w(z) - s + c)$ and in our case

$$-w(Q) - c + \sum_{k=2}^g w(x_k) = -s + c$$

and so

$$s = w(Q) + 2c - \sum_{k=2}^g w(x_k).$$

By definition of a complementary divisor we know that

$$\sum_{k=2}^g x_k + \sum_{k=2}^g y_k$$

is the divisor of a differential of the first kind. Let q be the element of \mathbb{C}^g modulo the periods so that $q \equiv \sum_{\ell=1}^{2g} w(u_\ell)$ when $\sum_{\ell=1}^{2g} u_\ell$ is the divisor of a differential of the first kind and the element q is independent of the choice of the differential of the first kind by Abel's theorem. So we can rewrite (*) as

$$-w(Q) + \sum_{k=1}^g w(z_k) - \sum_{k=2}^g w(y_k) \equiv 2c - q.$$

Consider the case when all y_2, \dots, y_g are distinct. Then we have seen above that

$$\{z_1, \dots, z_g\} = \{Q, y_2, \dots, y_g\}.$$

Hence we can conclude that $2c - q \equiv 0$. Now we go back to the general case. We have from $2c - q \equiv 0$ the conclusion that

$$-w(Q) + \sum_{k=1}^g w(z_k) - \sum_{k=2}^g w(y_k) \equiv 0.$$

Since we have assumed that $z_1 = Q$, we conclude that

$$\sum_{k=2}^g w(z_k) \equiv \sum_{k=2}^g w(y_k).$$

Since $\sum_{k=2}^g y_k$ is general, it follows that

$$\sum_{k=2}^g z_k \equiv \sum_{k=2}^g y_k.$$

This concludes the proof of the theorem.

Remark on the Geometric Meaning of the Point c in the Abelian Variety. A by-product of the above proof is a geometric meaning of the “offset” point c in the abelian variety $\mathbb{C}^g / (\mathbb{Z}^g + Z\mathbb{Z}^g)$. For $P_1, \dots, P_\ell \in M$, the point $\sum_{j=1}^{\ell} w(P_j)$ in $\mathbb{C}^g / (\mathbb{Z}^g + Z\mathbb{Z}^g)$ can be interpreted as the line bundle of zero

Chern class on M (i.e., divisor of degree 0 in M) represented by $P_1 + \cdots + P_\ell - \ell P_0$. In this interpretation the point c satisfies $q = 2c$, which means that the line bundle of zero Chern class on M (i.e., divisor of degree 0 in M) represented by $q - (2g - 2)P_0$ is equal to the line bundle of zero Chern class on M (i.e., divisor of degree 0 in M) represented by $c - (g - 1)P_0$. We can thus interpret c as the point representing a “square root” of the canonical line bundle of M .

Representation of Meromorphic Functions on General Riemann Surfaces as Pullbacks by Jacobian Maps of Quotients of Products of Translates of Theta Functions. We are now ready to use the preceding results to represent any meromorphic function on a compact Riemann surface as the pullback by the Jacobian map of the quotient of two products of translates of the theta function.

Theorem. Let $f(P)$ be a nonconstant meromorphic function on a compact Riemann surface M of genus $g \geq 2$. Let $a_1 + \cdots + a_m - b_1 - \cdots - b_m$ be the divisor of f . Let x_2, \cdots, x_g be a general divisor of degree $g - 1$ disjoint from the union of $\{a_1, \cdots, a_m\}$ and $\{b_1, \cdots, b_m\}$. Then

$$f(P) = \delta \prod_{\nu=1}^m \frac{\theta(w(P) - w(a_\nu) - c + \sum_{k=2}^g w(x_k))}{\theta(w(P) - w(b_\nu) - c + \sum_{k=2}^g w(x_k))}$$

for some constant δ when we choose the paths of integration so that

$$\sum_{k=1}^g w(a_k) = \sum_{k=1}^g w(b_k).$$

Proof. Let $y_2 + \cdots + y_g$ be the complementary divisor of $x_2 + \cdots + x_g$. From the above theorem we know that the divisor of

$$\theta \left(w(P) - w(a_\nu) - c + \sum_{k=2}^g w(x_k) \right)$$

as a function of P is $a_\nu + \sum_{k=2}^g y_k$. The condition

$$\sum_{k=1}^g w(a_k) = \sum_{k=1}^g w(b_k)$$

guarantees that every point in the period lattice of the Riemann surface is is a period of the function

$$\frac{\theta(s - w(a_\nu) - c + \sum_{k=2}^g w(x_k))}{\theta(s - w(b_\nu) - c + \sum_{k=2}^g w(x_k))}$$

as a function of the variable s in \mathbb{C}^n so that

$$\frac{\theta(w(P) - w(a_\nu) - c + \sum_{k=2}^g w(x_k))}{\theta(w(P) - w(b_\nu) - c + \sum_{k=2}^g w(x_k))}$$

is a meromorphic function on the Riemann surface. The divisor of

$$\prod_{\nu=1}^m \frac{\theta(w(P) - w(a_\nu) - c + \sum_{k=2}^g w(x_k))}{\theta(w(P) - w(b_\nu) - c + \sum_{k=2}^g w(x_k))}$$

is precisely

$$a_1 + \cdots + a_m - b_1 - \cdots - b_m$$

and we can conclude that the quotient of $f(P)$ and

$$\prod_{\nu=1}^m \frac{\theta(w(P) - w(a_\nu) - c + \sum_{k=2}^g w(x_k))}{\theta(w(P) - w(b_\nu) - c + \sum_{k=2}^g w(x_k))}$$

is a constant.