Width of Strip of all Zeta Function Zeroes and Error Term of Growth of Partial Sum of Dirichlet Series of Logarithmic Derivative of Zeta Function at the Origin

THEOREM. (a) Suppose there exists $0 < \theta < 1$ such that the zero-set Z of $\zeta(s)$ in $\{0 \leq \operatorname{Re} s \leq 1\}$ is contained in $\{1-\theta \leq \operatorname{Re} s \leq \theta\}$. Then

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^{\theta} (\log x)^2\right).$$

(b) Suppose that for some $0 < \alpha < 1$,

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^{\alpha}\right).$$

Then Z is contained in $\{1-\alpha \leq \operatorname{Re} s \leq \alpha\}$.

Proof. For x equal to a non-integer > 1 we use the explicit formula

$$(*) \qquad \sum_{n<x} \Lambda(n) = x - \sum_{\substack{\rho \in Z \\ |\operatorname{Im} \rho| \leq T}} \frac{x^{\rho}}{\rho} + O\left(\frac{x(\log x)^2}{T} + \frac{x \log x}{\langle x \rangle T} + \frac{x \log T}{T}\right),$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^e,\ e \in \mathbb{N}, e \geq 1, p \text{ prime} \\ 0 & \text{otherwise.} \end{cases}$$

Since the zero-set Z of $\zeta(s)$ in $\{0 \leq \operatorname{Re} s \leq 1\}$ is contained in $\{1-\theta \leq \operatorname{Re} s \leq \theta\}$, it follows that

$$\left| \sum_{\substack{\rho \in Z \\ |\operatorname{Im} \rho| \leq T}} \frac{x^{\rho}}{\rho} \right| \leq x^{\theta} \sum_{\substack{\rho \in Z \\ |\operatorname{Im} \rho| \leq T}} \frac{1}{|\rho|}.$$

Since there are $O(\log |t|)$ zeroes with $t \leq |\operatorname{Im} \rho| < t+1$,

$$\sum_{\substack{\rho \in Z \\ |\operatorname{Im} \rho| \leq T}} \frac{1}{|\rho|} \leq \sum_{j=1}^{T} \frac{\log j}{j} = O\left((\log T)^2\right).$$

We can now rewrite $(*)$ as

$(**)$

$$\sum_{n \leq x} \Lambda(n) = x - \sum_{\substack{\rho \in Z \\ |\operatorname{Im} \rho| \leq T}} \frac{x^{\rho}}{\rho} + O\left(x^{\theta} (\log T)^2 + \frac{x(\log x)^2}{T} + \frac{x \log x}{\langle x \rangle T} + \frac{x \log T}{T}\right).$$

To get (a), we set $T = x^{1-\theta}$ in $(**)$ to conclude that

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^{\theta} (\log x)^2\right).$$

For the proof of (b), we use summation by parts. Let $A(n) = \sum_{n \leq x} \Lambda(n)$. Then on $\{\operatorname{Re} s > 1\}$ since $A(n) = O(n)$,
(†)

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n>1} \frac{\Lambda(n)}{n^s} = \sum_{n>1} \frac{A(n) - A(n-1)}{n^s} = \sum_{n \geq 1} A(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right).$$

From the assumption

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x^{\alpha}\right),$$

we can write $A(n) = n + A_1(n)$ with $A_1(n) = O(n^{\alpha})$. Now (†) reads

$$(\dagger\dagger) \qquad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} A_1(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + \sum_{n \geq 1} n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right).$$

By summation by parts the second term on the right-hand side can be transformed to

$$\sum_{n \geq 1} n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) = \sum_{n \leq 1} \frac{n - (n-1)}{n^s} = \sum_{n \leq 1} \frac{1}{n^s} = \zeta(s).$$

Let $\sigma = \operatorname{Re} s$. By the Mean Value Theorem,

$$\frac{1}{n^s} - \frac{1}{(n+1)^s} = O\left(\frac{1}{n^{\sigma+1}}\right).$$

Hence

$$A_1(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) = O\left(\frac{1}{n^{\sigma+1-\alpha}}\right)$$

and the series

$$\sum_{n \geq 1} A_1(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right)$$

converges for $\sigma + 1 - \alpha > 1$ or $\sigma > \alpha$ and represents a holomorphic function there. Since $\zeta(s)$ is holomorphic on $\{\sigma > 0\}$ except for a pole at $s = 1$, it follows from (††) that $-\frac{\zeta'(s)}{\zeta(s)}$ is holomorphic on $\{\sigma > \alpha\}$ except for at a pole at $s = 1$. This means that Z is contained in $\{1 - \alpha \leq \operatorname{Re} s \leq \alpha\}$. Q.E.D.