

Counting of Zeroes of Riemann Zeta Function on Critical Strip

Let $N(T)$ be the number of zeroes of the function $\zeta(s)$ in

$$\{0 \leq \sigma \leq 1, 0 < t \leq T\}.$$

We would like to prove the following formula for the counting of the zeroes of the Riemann zeta function on the critical strip

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + O(\log T).$$

There are three ingredients in the proof. The first is Jensen's formula, the second is the argument principle, and the third is the growth rate of the argument of the Riemann zeta function on vertical lines.

If T is not the ordinate of a zero of $\zeta(s)$, let $S(T)$ denote the value of

$$\frac{\arg \zeta\left(\frac{1}{2} + iT\right)}{\pi}$$

obtained by continuous variation along the straight lines joining 2 , $2 + iT$, $\frac{1}{2} + iT$, starting with the value 0 . If T is the ordinate of a zero, let $S(T) = S(T + 0)$. Let

$$L(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + \frac{7}{8}.$$

We are going to verify that

$$(*) \quad N(T) = L(T) + S(T) + O\left(\frac{1}{T}\right).$$

The number of zeroes of the function

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right)$$

in the rectangle with vertices at

$$T - \frac{3}{2}i, \quad T + \frac{3}{2}i, \quad -T + \frac{3}{2}i, \quad -T - \frac{3}{2}i$$

is $2N(T)$, so that

$$2N(T) = \frac{1}{2\pi i} \int \frac{\Xi'(z)}{\Xi(z)} dz$$

with the integration along the rectangle in the positive sense. Since $\Xi(z)$ is even and real for real z , this is equal to

$$\begin{aligned} & \frac{2}{\pi i} \left(\int_T^{T+\frac{3}{2}i} \frac{\Xi'(z)}{\Xi(z)} dz + \int_{T+\frac{3}{2}i}^{\frac{3}{2}i} \frac{\Xi'(z)}{\Xi(z)} dz \right) \\ &= \frac{2}{\pi i} \left(\int_2^{2+iT} \frac{\xi'(z)}{\xi(z)} dz + \int_{2+iT}^{\frac{1}{2}+iT} \frac{\xi'(z)}{\xi(z)} dz \right) = \frac{2}{\pi} \Delta \arg \xi(s), \end{aligned}$$

where Δ denotes the variation from 2 to $2+iT$ and then to $\frac{1}{2}+iT$ along straight lines. From

$$\xi(s) = \frac{s(1-s)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

it follows that

$$\pi N(T) = \Delta \arg s(s-1) + \Delta \arg \pi^{-\frac{s}{2}} + \Delta \arg \Gamma\left(\frac{s}{2}\right) + \Delta \arg \zeta(s).$$

Now

$$\begin{aligned} \Delta \arg \Gamma\left(\frac{s}{2}\right) &= \operatorname{Im} \left(\log \Gamma\left(\frac{1}{4} + \frac{iT}{2}\right) \right) \\ &= \operatorname{Im} \left(\left(-\frac{1}{4} + \frac{iT}{2} \right) \log \left(\frac{iT}{2} \right) - \frac{iT}{2} + O\left(\frac{1}{T}\right) \right) \\ &= \frac{T}{2} \log \frac{T}{2} - \frac{\pi}{8} - \frac{T}{2} + O\left(\frac{1}{T}\right). \end{aligned}$$

Adding these results, we obtain the theorem, provided that T is not the ordinate of a zero. If T is the ordinate of a zero, the result follows from the definitions and what has already been proved, the term $O\left(\frac{1}{T}\right)$ being continuous. This finishes the verification of (*). To continue with the proof of the formula for the counting of the zeroes of the Riemann zeta function on the critical strip we need the following statement on the growth rate of the argument of the Riemann zeta function on vertical lines.

Growth Rate of Argument of Riemann Zeta Function on Vertical Line. Let $m > 0$ such that $|\operatorname{Re} \zeta(2+it)| \geq m$. Let $M_T > 0$ such that $|\zeta(\sigma+it)| \leq M_T$ for $\sigma \geq 0$ and $1 \leq t \leq T$. If T is not the ordinate of a zero of $\zeta(s)$, then

$$|\arg \zeta(\sigma+iT)| \leq \frac{\pi}{\log 3} \left(\log M_T + \log \frac{1}{m} \right) + \frac{3\pi}{2}$$

for $\sigma \geq \frac{1}{2}$.

We now verify the statement on the growth rate of the argument of the Riemann zeta function on vertical lines. Since $\arg \zeta(2) = 0$ and

$$\arg \zeta(s) = \arctan \left(\frac{\operatorname{Im} \zeta(s)}{\operatorname{Re} \zeta(s)} \right),$$

where $\operatorname{Re} \zeta(s)$ does not vanish on $\sigma = 2$, we have

$$|\arg \zeta(2 + it)| < \frac{\pi}{2}.$$

If $\operatorname{Re} \zeta(s)$ vanishes q times between $2 + iT$ and $\frac{1}{2} + iT$, this interval is divided into $q + 1$ parts, throughout each of which $\operatorname{Re} \zeta(s) \geq 0$ or $\operatorname{Re} \zeta(s) \leq 0$. Hence in each part the variation of $\arg \zeta(s)$ does not exceed π . Hence

$$|\arg \zeta(s)| \leq \left(q + \frac{3}{2} \right) \pi \quad \text{for } \sigma \geq \frac{1}{2}.$$

Now q is the number of zeroes of the function

$$g(z) = \frac{1}{2} (\zeta(z + iT) + \zeta(z - iT))$$

for $\operatorname{Im} z = 0$ and $\frac{1}{2} \leq \operatorname{Re} z \leq 2$. Hence $q \leq n\left(\frac{3}{2}\right)$, where $n(r)$ denotes the number of zeroes of $g(z)$ for $|z - 2| \leq r$. Also

$$\int_0^2 \frac{n(r)}{r} dr \geq \int_{\frac{3}{2}}^2 \frac{n(r)}{r} dr \geq n\left(\frac{3}{2}\right) \log \frac{4}{3},$$

and by Jensen's theorem

$$\begin{aligned} \int_0^2 \frac{n(r)}{r} dr &= \frac{1}{2\pi} \int_0^{2\pi} \log |g(2 + 2e^{i\theta})| d\theta - \log |g(2)| \\ &\leq \log M_T + \log \frac{1}{m}. \end{aligned}$$

This finishes the verification of the statement on the growth rate of the argument of the Riemann zeta function on vertical lines. Since we have the following growth rate for $\zeta(s)$ on vertical lines

$$\zeta(s) = O(|t|^A),$$

where A is a generic symbol for a positive constant, it follows that $S(T) = O(\log T)$ and (*) follows.